Math 318 – homework 5 – due 2023-03-03

Problem 1. Let $X_1, X_2, \dots, X_n$ be independent random variables, each with uniform distribution on $[0,1]$. Let M be the minimum of these random variables.

(a) Find the cumulative distribution function F_M of the random variable M. (Hint: It might be easier to find $1 - F_M$.)
(b) Find the probability density function of M.
(c) Determine the mean and variance of M.
(d) Let $Y = n \cdot M$. Find the probability density function of Y.
(e) Find the limit as $n \to \infty$ of the pdf of Y.

Problem 2. Let X be uniform on $[0,1]$ and Y be uniform on $[-1,0]$. Compute $\mathrm{Cov}(X, X^2)$ and $\mathrm{Cov}(Y, Y^2)$. (The first is positive and the second is negative, consistent with the fact that X^2 increases when X whereas the opposite is true for Y.)

Problem 3. Let X and Y be independent $N(0,1)$ random variables. Let $U = X + Y$ and $V = X - Y$.

(a) Find the joint probability density function of U and V.
(b) Show that U and V are independent.
(c) What is the marginal distribution of U?

Problem 4. Let X, Y be the length of time a machine works before breaking and the length of time it takes to fix it, and let $Z = X + Y$. Suppose X, Z have the joint pdf

$$f_{X,Z}(x,z) = \lambda^2 e^{-\lambda z} 1_{0 \le x \le z}.$$

(a) Find the density of X.
(b) Find the density of Z.
(c) Find the joint density of X, Y. (Hint: The joint density of X, Y is $\frac{d}{dx}\frac{d}{dy}\mathbb{P}(X \le x, Y \le y)$.)
(d) Find the density of Y.

Problem 5. Consider a random variable $X \sim \mathrm{Bin}(100, 0.6)$. We are interested in the event $A = \{X \le 50\}$.

(a) Calculate in python $P(A)$ exactly.
(b) Use the CLT (writing X is a sum of 100 Bernoulli variables) to argue that $Z = \frac{X-a}{b}$ is approximately $N(0,1)$ for some a, b.
(c) Use this to estimate $P(A) = P(Z \le t) = \Phi(t)$ for some t, and calculate this numerically.
(d) The event A is the same as $\{X < 51\}$. Write this as $\{Z < t'\}$ and calculate $\Phi(t')$.
(e) We may get a better approximation by taking the median of these two. Convert $A = \{X \le 50.5\}$ to a statement on Z and the normal approximation for that.

Problem 6. Suppose that X, Y are independent discrete random variables taking values in $\mathbb{N}$, with p.m.f.'s p_X and p_Y, respectively. We have seen in class that the random variable $X + Y$ has p.m.f. $p_X * p_Y$, the convolution of p_X and p_Y, i.e.,

$$p_{X+Y}(k) = (p_X * p_Y)(k) = \sum_{j \ge 0}^{k} p_X(j) p_Y(k-j).$$

(a) Let $X_1, X_2, \dots$ be i.i.d. (independent and identically distributed) random variables that are uniform over the set $\{0,1,2\}$, and let $S_n = X_1 + \cdots + X_n$. Calculate (in python) and graph the p.m.f.'s of S_n for $n = 1, 2, 3, 4, 5, 10, 50$. Superimpose each of these with the pdf of a normal random variable with the same mean and variance as S_n.

(b) Let $Y_1, Y_2, \ldots$ be i.i.d. random variables such that

$$\begin{aligned} P(Y_i = -1) = P(Y_i = 0) &= 1/15; \\ P(Y_i = 1) &= 11/15; \\ P(Y_i = 2) = P(Y_i = 4) &= 1/15. \end{aligned}$$

Let $T_n = Y_1 + \cdots + Y_n$. Calculate and graph the p.m.f.'s of T_n for the same n's. Superimpose each of these with the pdf of a normal random variable with the same mean and variance as T_n.

Extra practice problems

Ross, chapter 2: 53,58,63,71,72,76