Math 318 – homework 7 – due 2023-03-17

Problem 1. (a) If X is an integer valued random variable, show that the characteristic function of X has period 2π.

(b) Prove the converse: if $\phi(t) = \phi(t + 2\pi)$ for every t, then show that X takes only integer values. (Hint: If a random variable Y satisfies $Y \geq 0$ and $E[Y] = 0$ then $P(Y = 0) = 1$. Use this for a carefully chosen function of X. Hint: consider $\phi(2\pi)$.)

Problem 2. Let U, V be two independent random variables, with $E[U] = E[V] = 0$, and let $X = U + V$. Assume also that U and V have finite moments. Find an expression for $E[X^3]$ and for $E[X^4]$, in terms of moments of U, V.

Problem 3. Let A be an $n \times n$ matrix, and let $X = (X_1, \dots, X_n)^T$ be a vector of i.i.d. $N(0, 1)$ random variables. Let $Y = AX$, and Y_i its coordinates.

(a) What is the distribution of Y_i?

(b) Find $\mathrm{Cov}(Y_i, Y_j)$.

(c) If $n = 2$ and A is invertible, show that Y_1, Y_2 have joint probability density function

$$\frac{1}{(2\pi)^{n/2} |\det A|} e^{-(y^T A^{T^{-1}} A^{-1} y)/2}$$

on $\mathbb{R}^2$, where $y = (y_1, y_2)^T$ is a vector.) (Hint: What is the Jacobian of the mapping from X to Y?)

bonus For any n, if A is invertible, show that Y has probability density function with the same formula.

Problem 4. Cosider the random walk in one dimension $\mathbb{Z}$ with bias. We will consider $p = 0.5$, $p = 0.51$, and $p = 0.502$. Let $X_i = \pm 1$ with $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p$ be independent. Let $S_n = X_1 + X_2 + \cdots + X_n$. For each of these, do the following.

(a) Simulate and plot a random walk with 10^6 steps. Submit your code and a plot of $S_0, \dots, S_n$.

(b) Simulate 1000 independent random walks, each for up to 10^6 steps. For each walk, let $T > 0$ be the first time the walk returns to 0. If a walk does not return to 0, let $T = 10^6$. For each of the p's, how many of the 1000 walks did not return to 0? Submit histograms of the values of T observed.

(c) Make a log-log plot of the fraction of times $T > k$ for $k = 0, \dots, 10^6$.

(d) Based on the previous plot, guess the asymptotics of $P(T > n)$ as $n \to \infty$. What do you think is the mean of T if there was no limit of 10^6?

Problem 5. This problem investigates the recurrence of 2-dimensional random walk. Let (X_n, Y_n) be a two dimensional random walk. At each step pick either X or Y and either increase or decrease it by 1, all with equal probabilities, and independeent of other steps.

(a) Simulate this process for 10^6 steps. Submit code and a plot of the walk in the $X - Y$ plane (there is no coordinate for time, though you could use colour to indicate the time a point is visited.

(b) Simulate 1000 random walks for up to 10^6 steps, and for each keep track of the number of steps before it returns to 0 for the first time. How many of the walks failed to return to 0? Is this consistent with the theorem that the random walk is recurrent?

(c) Make a histogram of the resulting return times for the walks that did return, on a logarithmic scale. Can you guess how fast $P(T > n)$ decays from this?

Note. it is much faster to only generate each random walk until it returns to 0, and stop there. Otherwise this make take a long time to run.

Extra practice problems

(a) Chapter 4: 23,24,30,35

(b) The Polya urn: An urn has initially one red and one green ball. At each step we draw uniformly a ball from the urn, return it, and add oe more ball of the same colour. Let (G_n, R_n) be thenumber of green/red balls when the total is n, so that initally $G_2 = R_2 = 1$.

(i) Prove that G_n is uniformly distributed on $\{1, \dots, n-1\}$.

(ii) Find the probability that the balls $n+1$ and $n+2$ are the same colour.

(iii) Let X_n be the fraction of red balls when there are n balls in the urn. Prove that $E(X_{n+1}|X_n) = X_n$. (Such a process is called a martingale.

(iv) Simulate this several times and plot the fraction of red balls over time. Note: The martingale convergence theorem states that a bounded martingale converges.