Math 318 – homework 8 – due 2023-03-24

Problem 1. (a) Consider a walk in $\mathbb{Z}$ that takes steps 0 or ± 2, such that the probability of step 2 equals the probability of step -2. Show that this walk is recurrent. (Hint: use the fact that the simple random walk on $\mathbb{Z}$ is recurrent.)

(b) Use this to show that if X_n and Y_n are two random walks in $\mathbb{Z}$ with step 1 with probability p and -1 with probability $1-p$, then there are infinitely many n's for which $X_n = Y_n$. (Hint: what is $X_n - Y_n$?)

Problem 2. In each of (a)–(d), determine whether or not the given Markov chain is irreducible, and identify the communicating classes. For each state, determine whether it is recurrent or transient, and periodic or aperiodic. In (a) and (b), the state space is $S = \{1, \dots, 5\}$.

(a) $$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) $$P = \begin{pmatrix} 0 & \frac{4}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) The simple symmetric random walk on $\mathbb{Z}^d$ (answer for each $d \geq 1$).

(d) The random walk on $\mathbb{Z}$ with probability 1/3 of moving right and 2/3 of moving left.

Problem 3. Initially, Anthony has n camels, and Cleopatra has n horses. Every day, each of them selects one of their animals at random, and sends it to the other as a gift. Let X_m denote the number of camels that Cleopatra has urn after m days.

(a) Find the transition matrix for this chain.

(b) Show that the stationary distribution for this chain is $\pi_i = \frac{\binom{n}{i}^2}{\binom{2n}{n}}$: If X_0 has this distribution then so does X_1.

Hint: Do not trust Greeks bearing gifts.

Problem 4. There are n coins on the table. Each step we choose at random one of the coins and toss it again. Let X_m be the number of heads showing. Show that this chain has transition probabilities

$$P_{ii} = \frac{1}{2}, \quad P_{i,i-1} = \frac{i}{2n}, \quad P_{i,i+1} = \frac{n-i}{2n}.$$

Bonus: Guess what the stationary distribution is, and verify your guess.

Problem 5. (a) For the gambler's ruin problem, let M_k denote the expected number of bets that will be made if the player initially has \$k, and stops at 0 or n. Let p be the probability of winning each bet, and $q = 1 - p$. Show that $M_0 = M_n = 0$ and

$$M_k = 1 + pM_{k+1} + qM_{k-1}$$

for $0 < k < n$. (Hint: Compute the expectation of the number of bets X by conditioning on the outcome of the first game. If A is the event that the player wins the first bet,

$$EX = E[X|A]P(A) + E[X|A^c]P(A^c).$$

(b) Solve the equations in (a) to obtain

$$M_k = k(n-k) \text{ if } p = 1/2,$$

and

$$M_k = \frac{k}{q-p} - \frac{n}{q-p}\frac{1-\alpha^k}{1-\alpha^n} \text{ if } p = 1/2,$$

where $\alpha = q/p$. To do this, proceed as follows. First, find the general solution to the homogeneous equation $M_k = pM_{k+1} + qM_{k-1}$ (as done in class). Next, find a particular solution to the inhomogeneous equation $M_k = 1 + pM_{k+1} + qM_{k-1}$ (try $M_k = ck^2$ for $p = 1/2$ and $M_k = ck$ for $p \neq 1/2$). Add the general solution of the homogeneous equation to the particular solution of the inhomogeneous equation. Finally, solve for the two unknown constants in the general solution by using the boundary conditions.

Problem 6. Simulate the Anthony and Cleopatra process with $n = 1000$. Run the process for 20000 steps. Submit a plot of X_t over time. Additionally, keep track of how many times each state is visited, from time 0 to time 1000, from 1000–2000, and finally, in times 10000-20000, and generate histograms for these. Submit your code.

Extra practice problems

(a) Chapter 4: 2,13,14,15,38,57

(b) Write down a 6×6 stochastic matrix and determine its irreducible classes, recurrence and periodicity of states.

(c) Write down a 4 state irreducible markov chain with $P_{ii} = 0$ for all i and find its stationary distribution.

(d) A matrix is doubly stochastic if it is stochastic and every columns also adds up to 1. Show that if P is doubly stochastic then the stationary distribution is $\pi_i = 1/n$ where n is the number of states.

(e) Find the stationary distribution of the following Markov chain. The states are $0, 1, 2, \ldots, N$. From each i except 0 and N the walk moves with equal probabilities to $i \pm 1$. From 0 the walk moves always to 1. From N the walk moves with equal probabilities to 0 and $N - 1$.