Math 318 – homework 8 – due 2023-03-24

Problem 1. (a) Consider a walk in $\mathbb{Z}$ that takes steps 0 or ± 2, such that the probability of step 2 equals the probability of step -2. Show that this walk is recurrent. (Hint: use the fact that the simple random walk on $\mathbb{Z}$ is recurrent.)

(b) Use this to show that if X_n and Y_n are two random walks in $\mathbb{Z}$ with step 1 with probability p and -1 with probability $1-p$, then there are infinitely many n's for which $X_n = Y_n$. (Hint: what is $X_n - Y_n$?)

Problem 2. (a) The steps of size 0 just change the time at which the walk reaches values, and slows it down, so we can consider the random walk that takes only steps of ± 2. This is just double the regular random walk on the line, and we know this is recurrent if the probability of +1 is equal to the probability of -1, and transient if not.

(b) Consider the proces $Z_n = X_n - Y_n$. Since X and Y change by ± 1, the increment of Z_n must be one of $\{0, 2, -2\}$. Specifically, $P(Z_{n+1} = Z_n) = p^2 + (1-p)^2$ and $P(Z_{n+1} = Z_n \pm 2) = p(1-p)$ each. Since the probability of $+2$ equals the probability of -2, the walk Z_n is recurrent from part (a). This implies that $Z_n = 0$ infinitely often, which is to say $X_n = Y_n$.

Problem 3. In each of (a)–(d), determine whether or not the given Markov chain is irreducible, and identify the communicating classes. For each state, determine whether it is recurrent or transient, and periodic or aperiodic. In (a) and (b), the state space is $S = \{1, \ldots, 5\}$.

(a) $$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) $$P = \begin{pmatrix} 0 & \frac{4}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) The simple symmetric random walk on $\mathbb{Z}^d$ (answer for each $d \geq 1$).

(d) The random walk on $\mathbb{Z}$ with probability 1/3 of moving right and 2/3 of moving left.

Solution.

(a) The communicating clases are $\{1\}$:transient;
$\{2, 3\}$: recurrent, period 2;
$\{4, 5\}$: recurrent, aperiodic.

(b) The chain is irreducible, aperiodic and recurrent.

(c) This is irreducible with period 2. Recurrent if $d = 1, 2$ and transient for $d > 2$.

(d) This is irreducible with period 2 and transient.

Problem 4. Initially, Anthony has n camels, and Cleopatra has n horses. Every day, each of them selects one of their animals at random, and sends it to the other as a gift. Let X_m denote the number of camels that Cleopatra has after m days.

(a) Find the transition matrix for this chain.

(b) Show that the stationary distribution for this chain is $\pi_i = \frac{\binom{n}{i}^2}{\binom{2n}{n}}$: If X_0 has this distribution then so does X_1.

Hint: Do not trust Greeks bearing gifts.

Problem 5. (a) If Cleopatra has k camels, then Anthony has $n-k$ camels. In that case, the probability that Anthony sends a camel and Cleopatra a horse is $\frac{n-k}{n} \cdot \frac{n-k}{n}$. We therefore have $p_{k,k+1} = \frac{(n-k)^2}{n^2}$. Similarly, $p_{k,k-1} = \frac{k^2}{n^2}$ and $p_{k,k} = \frac{2k(n-k)}{n^2}$ (since they either both send camels or both horses).

(b) This chain is reversible w.r.t. this π. We see this by verifying

$$\pi_k p_{k,k+1} = \pi_{k+1} p_{k+1,k}.$$

Since it is reversible w.r.t. π, this is the stationary distribution. Writing out the factorials, this is easy to check.

Alternatively, we can check that

$$\pi_k = \pi_k p_{k,k} + \pi_{k+1} p_{k+1,k} + \pi_{k-1} p_{k-1,k}.$$

Problem 6. There are n coins on the table. Each step we choose at random one of the coins and toss it again. Let X_m be the number of heads showing. Show that this chain has transition probabilities

$$P_{ii} = \frac{1}{2}, \quad P_{i,i-1} = \frac{i}{2n}, \quad P_{i,i+1} = \frac{n-i}{2n}.$$

Bonus: Guess what the stationary distribution is, and verify your guess.

Solution. We stay at i if the coin we toss comes up whatever it showed before, so with probability 1/2. If we pick a con showing heads (probability i/n) and it comes up tails (probability 1/2), then we decrease by 1. Increasing by 1 is similar.

After every coin has been tossed at least once, they are all independent, so the number of heads is binomial $\text{Bin}(n, 1/2)$.

Problem 7. (a) For the gambler's ruin problem, let M_k denote the expected number of bets that will be made if the player initially has \$k, and stops at 0 or n. Let p be the probability of winning each bet, and $q = 1-p$. Show that $M_0 = M_n = 0$ and

$$M_k = 1 + pM_{k+1} + qM_{k-1}$$

for $0 < k < n$. (Hint: Compute the expectation of the number of bets X by conditioning on the outcome of the first game. If A is the event that the player wins the first bet,

$$EX = E[X|A]P(A) + E[X|A^c]P(A^c).$$

(b) Solve the equations in (a) to obtain

$$M_k = k(n-k) \text{ if } p = 1/2,$$

and

$$M_k = \frac{k}{q-p} - \frac{n}{q-p}\frac{1-\alpha^k}{1-\alpha^n} \text{ if } p = 1/2,$$

where $\alpha = q/p$. To do this, proceed as follows. First, find the general solution to the homogeneous equation $M_k = pM_{k+1} + qM_{k-1}$ (as done in class). Next, find a particular solution to the inhomogeneous equation $M_k = 1 + pM_{k+1} + qM_{k-1}$ (try $M_k = ck^2$ for $p = 1/2$ and $M_k = ck$ for $p \neq 1/2$). Add the general solution of the homogeneous equation to the particular solution of the inhomogeneous equation. Finally, solve for the two unknown constants in the general solution by using the boundary conditions.

Problem 8. Simulate the Anthony and Cleopatra process with $n = 1000$. Run the process for 20000 steps. Submit a plot of X_t over time. Additionally, keep track of how many times each state is visited, from time 0 to time 1000, from 1000–2000, and finally, in times 10000-20000, and generate histograms for these. Submit your code.

Extra practice problems

(a) Chapter 4: 2,13,14,15,38,57

(b) Write down a 6×6 stochastic matrix and determine its irreducible classes, recurrence and periodicity of states.

(c) Write down a 4 state irreducible markov chain with $P_{ii} = 0$ for all i and find its stationary distribution.

(d) A matrix is doubly stochastic if it is stochastic and every columns also adds up to 1. Show that if P is doubly stochastic then the stationary distribution is $\pi_i = 1/n$ where n is the number of states.

(e) Find the stationary distribution of the following Markov chain. The states are $0, 1, 2, \ldots, N$. From each i except 0 and N the walk moves with equal probabilities to $i \pm 1$. From 0 the walk moves always to 1. From N the walk moves with equal probabilities to 0 and $N - 1$.