Math 318 – homework 9 solutions

Problem 1. Psmith has three books A,B,C which he keeps on a shelf After he reads from a book, he puts it at the front of the shelf. His favourite is A, which he selects with probability 2/3. He selects B with probability 1/6, and he selects C also with probability 1/6. This defines a Markov chain, with the state of the system given by the different possible orders for the books. For example, if the order was ACB and Psmith picked A, the order is unchanged. If he picks C it becomes CAB. In your solution, order your states in the order: ABC,ACB,BAC,BCA,CAB,CBA.

(a) Write down the transition matrix for the Markov chain.
(b) Determine the stationary distribution of the Markov chain.
(c) Suppose the books are now in order BCA. How many steps will it take, on average, until the books are again in the same order?

Solution.

(a) The transition matrix is

	ABC	ACB	BAC	BCA	CAB	CBA
ABC	2/3	0	1/6	0	1/6	0
ACB	0	2/3	1/6	0	1/6	0
BAC	2/3	0	1/6	0	0	1/6
BCA	2/3	0	0	1/6	0	1/6
CAB	0	2/3	0	1/6	1/6	0
CBA	0	2/3	0	1/6	0	1/6

(b) Using $\pi P = \pi$ and $\sum \pi_x = 1$ we can solve to find

$$\pi_{ABC} = \pi_{ACB} = \frac{1}{3} \qquad \pi_{BAC} = \pi_{CAB} = \frac{2}{15} \qquad \pi_{BCA} = \pi_{CBA} = \frac{1}{30}.$$

This can also be found by hand: using the symmetry between B and C we get that the three pairs of values are equal. The probability that the last record picked is A is 2/3, so $\pi_{ABC} = \pi_{ACB} = \frac{1}{3}$. The probability that the last record picked is B is 1/6, so $\pi_{BAC} + \pi_{BCA} = \frac{1}{6}$. Using one row from $\pi P = \pi$ gives an extra equation to find the values.

(c) The average time from x to return to x is $1/\pi_x$, which in this case is 30.

Problem 2. Tlaloc has 4 umbrellas, each either at home or at work. Each time he goes to work or back, it rains with probability p, independently of all other times. If it rains **and** there is at least one umbrella with him, he takes it. Otherwise, he gets wet. Let X_n be the number of umbrellas at his current location after n trips (so n even corresponds to home and n odd to work.

(a) Find the transition probabilities for the Markov chain X_n.
(b) Show that it is reversible and find the stationary distribution.
(c) What is the probability that Tlaloc gets wet on any given trip?
(d) What value of p maximizes this probability?
(e) What happens if he has N umbrellas for N other than 4?

Solution.

(a) With states in order $0, 4, 1, 3, 2$ we get the transition matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & p & 1-p \\ 0 & 0 & p & 1-p & 0 \\ 0 & p & 1-p & 0 & 0 \\ p & 1-p & 0 & 0 & 0 \end{pmatrix}$$

(b) The detailed balance condition gives $\pi_0 = (1-p)\pi_4$ and $\pi_4 = \pi_1 = \pi_3 = \pi_2$. Thus we can normalize to find

$$\pi = \left(\frac{1-p}{5-p}, \frac{1}{5-p}, \frac{1}{5-p}, \frac{1}{5-p}, \frac{1}{5-p} \right).$$

(c) The probability of getting wet is $p\pi_0 = \frac{p(1-p)}{5-p}$.

(d) Taking derivatives, we find this is maximized at $p = 5 - \sqrt{20} \approx 0.53$, and the probability is $9 - \sqrt{80} \approx 0.056$.

(e) In a similar way we can get $\pi_0 = \frac{1-p}{N+1-p}$ and $\pi_i = \frac{1}{N+1-p}$ for all other i. Therefore the probability of getting wet is $\frac{p(1-p)}{N+1-p}$, which is maximized at $p = N + 1 - \sqrt{N^2+N}$ with value $2N + 1 - 2\sqrt{N^2+N}$. Since $\sqrt{N^2+N} = N + 1/2 - 1/8N + O(N^{-2})$, the p that maximizes the probability of getting wet is $1/2 + O(1/N)$ and the probability is $1/(4N) + O(N^{-2})$.

Problem 3. Consider the random walk on $\mathbb{Z}$ with jump probabilities from n given by $P_{n,n+1} = p > 1/2$ and $P_{n,n-1} = 1 - p$. Find the probability that, starting at 0, the walk returns to 0 at some later time. One method: Let q be the probability that if we are at n we ever hit $n-1$. Argue why this does not depend on n. Condition on the first step from n and find an equation satisfied by q. Use the value of q to solve the original problem.

Problem 4. Consider the following (inefficient) way to shuffle a deck of cards. We pick randomly two distinct numbers i, j from 1–52, and swap the ith card and jth card. Show that the stationary distribution for this random walk is the uniform distribution over all 52! permutations. (Hint: this is random walk on a suitable graph.)

Problem 5. Simulate the previous method for shuffling a deck of cards. Start with the cards in order $[0, 1, \ldots, 51]$. Let F_n be the number of cards i that are in position i at time n. Simulate this process 1000 times, each time running for 200 steps, and each time restarting with an ordered deck.

(a) Plot the average value of F_n for each $n \leq 200$.

(b) What is the expectation of F at stationarity?

Extra practice problems

(a) Chapter 4: 2,13,14,15,38,57

(b) Write down a 6×6 stochastic matrix and determine its irreducible classes, recurrence and periodicity of states.

(c) Write down a 4 state irreducible markov chain with $P_{ii} = 0$ for all i and find its stationary distribution.

(d) A matrix is doubly stochastic if it is stochastic and every columns also adds up to 1. Show that if P is doubly stochastic then the stationary distribution is $\pi_i = 1/n$ where n is the number of states.

(e) Find the stationary distribution of the following Markov chain. The states are $0, 1, 2, \ldots, N$. From each i except 0 and N the walk moves with equal probabilities to $i \pm 1$. From 0 the walk moves always to 1. From N the walk moves with equal probabilities to 0 and $N-1$.