MATH 318 midterm

February 2023

1. 12 marks A random variable X is uniform in $\{1, 2, 3, 4, 10\}$.
   (a) What is the expectation $E[X]$?
   (b) What is the variance $\mathrm{Var}(X)$?
   (c) What is the characteristic function of X?

Solution:

(a) $E[X] = \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} + \frac{10}{5} = 4$.
(b) $E[X^2] = \frac{1}{5}(1^2 + 2^2 + 3^2 + 4^2 + 10^2) = 26$ so $\mathrm{Var}(X) = \mathbb{E}[X^2] - (EX)^2 = 26 - 4^2 = 10$.
(c) $\phi(t) = Ee^{itX} = \frac{1}{5}(e^{it} + e^{2it} + +e^{3it} + e^{4it} + e^{10it})$.

Problem 1.

2. 12 marks We toss five dice (all 6-sided).
   (a) What is the probability of getting exactly three 1s?
   (b) What is the probability of getting five different numbers?

   Next, we take four cards from a standard deck (13 cards in each of 4 suits).
   (c) What is the probability of getting exactly three hearts?
   (d) What is the probability of getting one card from each suit?

Solution:

(a) Binomial distribution gives $P = \binom{5}{3}(1/6)^3(5/6)^2$.
(b) $P = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6^5}$.
(c) $P = \frac{\binom{13}{3}\binom{39}{1}}{\binom{52}{4}}$.
(d) $P = \frac{13^4}{\binom{52}{4}}$.

3. 8 marks Let $X \sim N(0, 1)$ be a standard normal random variable, and let $Y = X^2$.
   (a) Write the cdf of Y in terms of Φ, where $\Phi(t) = P(X \le t)$ is the normal cdf.
   (b) Find a formula for the pdf of Y.

Solution:

(a) $F(t) = P(Y \le t) = P(|X| \le \sqrt{t}) = \Phi(\sqrt{t}) - \Phi(-\sqrt{t})$.
(b) Using $\Phi'(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ and the chain rule:

$$f(t) = \frac{\Phi'(\sqrt{t})}{2\sqrt{t}} + \frac{\Phi'(-\sqrt{t})}{2\sqrt{t}} = \frac{e^{-t/2}}{\sqrt{2\pi} \cdot 2\sqrt{t}} + \frac{e^{-t/2}}{\sqrt{2\pi} \cdot 2\sqrt{t}} = \frac{e^{-t/2}}{\sqrt{2\pi t}}$$

for $t > 0$.

4. 8 marks The king has three urns, each with some balls. The first (A) contains 1 white and 1 black ball. The second (B) contains 2 white and 3 black balls. The third (C) contains 4 white and 1 black balls. Each prisoner picks a random urn, and takes out a ball at random without looking. If they pick a black ball, they will be hanged. If they pick a white ball, they will be drowned.

   (a) What is the probability that a prisoner is hanged?
   (b) A prisoner ends up with a black ball. What is the conditional probability that they picked urn A?

Solution:

(a) Let A, B, C be the event of picking that urn, and H the event of hanging.

$$P(H) = P(H|A)P(A) + P(H|B)P(B) + P(H|C)P(C) = \frac{1}{2}\frac{1}{3} + \frac{3}{5}\frac{1}{3} + \frac{1}{5}\frac{1}{3} = \frac{13}{30}.$$

(b) Using Bayes:

$$P(A|H) = \frac{P(A)P(H|A)}{P(H|A)P(A) + P(H|B)P(B) + P(H|C)P(C)} = \frac{\frac{1}{2}\frac{1}{3}}{\frac{13}{30}} = \frac{5}{13}.$$

5. 14 marks Random variables X, Y have joint pdf $f(x, y) = x + 2y^3$ on $[0, 1] \times [0, 1]$ and 0 outside that square.

   (a) What is the marginal distribution of X?
   (b) What is the marginal distribution of Y?
   (c) Are they independent? (justify!)
   (d) What is $\text{Cov}(X, Y)$? (Write the needed integrals; compute them if you have time.)

Solution:

(a) $f_X(x) = \int_0^1 x + 2y^3 dy = x + \frac{1}{2}$.
(b) $f_Y(x) = \int_0^1 x + 2y^3 dx = 2y^3 + \frac{1}{2}$.
(c) No. $f(x, y) f_X(x) f_Y(y)$.
(d) $\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$. Integral below are is on the square:

$$E[XY] = \iint xy(x + 2y^3)dxdy = \frac{1}{6} + \frac{1}{5}.$$

Also $E[X] = \int_0^1 x(x + \frac{1}{2})dx = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ and $E[Y] = \int_0^1 y(2y^3 + \frac{1}{2})dy = \frac{2}{5} + \frac{1}{4} = \frac{13}{20}$. Combined,

$$\text{Cov}(X, Y) = \frac{1}{6} + \frac{1}{5} - \frac{7}{12}\frac{13}{20} = -\frac{1}{80}.$$