2025-01-15

Birthday problem: ~~$\star$~~ n people, birthdays uniform in $\{1 \ldots 365\}$.

$E = \{$ some pair have same birthday $\}$

$E^c =$ complement.

$S =$ sample space $= \{1 \ldots 365\}^n$ $\quad (x_1, x_2, \ldots, x_n)$

$$|E^c| = 365 \cdot 364 \cdot 363 \cdots (365-(n-1))$$

so $$P(E_c) = \frac{|E^c|}{|S|} = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \left(1 - \frac{n-1}{365}\right)$$

$$= \prod_{i=0}^{n-1} \left(1 - \frac{i}{365}\right)$$

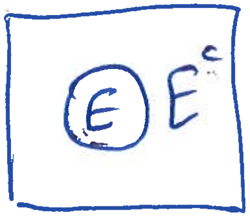
$P(E^c) = 0.49\ldots$ for $n=23$

$P(E^c) < 10^{-6}$ for $n=100$

$$P(E^c) = \exp\left(\sum_{i=0}^{n-1} \log\left(1-\frac{i}{365}\right)\right) \approx \exp\left(\sum_{i=0}^{n-1} -\frac{i}{365}\right) = e^{-\frac{n(n-1)}{2\cdot 365}}$$

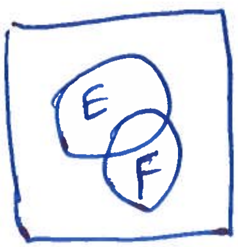
[For small x, $\log(1+x) \approx x$]

Properties of probability



$$P(E) + P(E^c) = P(S) = 1$$

so $P(E^c) = 1 - P(E)$

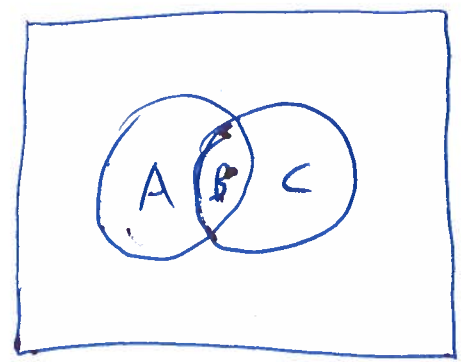


Claim: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

[inclusion-exclusion for 2 events]

$E = A \cup B$ $\quad P(E) = P(A) + P(B)$

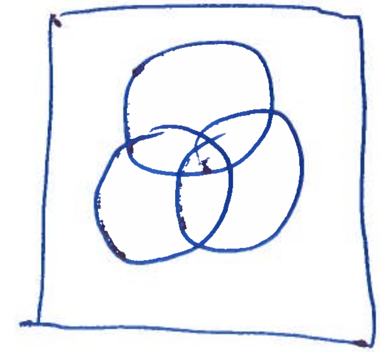
$F = B \cup C$ $\quad P(F) = P(B) + P(C)$



$E \cup F = A \cup B \cup C$

$$P(E \cup F) = P(A) + P(B) + P(C) = P(E) + P(F) - P(B)$$

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G)$$



Inclusion-Excl. for 3 events

For events $E_1, \ldots, E_n$:

$$P(E_1 \cup \ldots \cup E_n) = \sum_i P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) + \ldots \pm (-1)^{n-1} P(E_1 \cap E_2 \ldots \cap E_n)$$

n people take names out of the hat. [Secret-Santa]

what is $P(\underbrace{\text{no one gets their own name}}_{E})$

S = permutations of $1 \dots n$

$E_i = \{ i \text{ gets name } i \}$

want $P(E_1 \cup E_2 \cdots \cup E_n)$

$P(E_i) = \frac{1}{n}$

$$P\left(\bigcup_{i=1}^{n} E_i\right) = n \cdot \underbrace{P(E_1)}_{\frac{1}{n}} - \binom{n}{2} P(E_1 \cap E_2) + \binom{n}{3} P(E_1 \cap E_2 \cap E_3) - \binom{n}{4} P(\quad) \cdots$$

$$= n \cdot \frac{1}{n} + \binom{n}{2} \cdot \frac{(n-2)!}{n!} + \binom{n}{3} \frac{(n-3)!}{n!} - \binom{n}{4} \frac{(n-4)!}{n!} + \cdots - \pm \binom{n}{n} \frac{0!}{n!}$$

$|S| = n!$

$|E_1 \cap E_2| = 1 \cdot 1 \cdot (n-2)(n-3) \cdots 2 \cdot 1 = (n-2)!$

$P(E_1 \cap E_2) = \frac{(n-2)!}{n!} = P(E_i \cap E_j)$ for any $i<j$

$P(E_1 \cap E_2 \cap E_3) = \frac{(n-3)!}{n!}$

for k people: $\frac{(n-k)!}{n!}$

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \frac{(n-i)!}{n!}$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{\cancel{n!}}{i!\,\cancel{(n-i)!}} \cdot \frac{\cancel{(n-i)!}}{\cancel{n!}} = \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{1}{i!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots \pm \frac{1}{n!}$$

$$\approx 1 - e^{-1} \approx 0.63\dots$$

$$P(\text{no one gets own name}) \approx e^{-1}$$