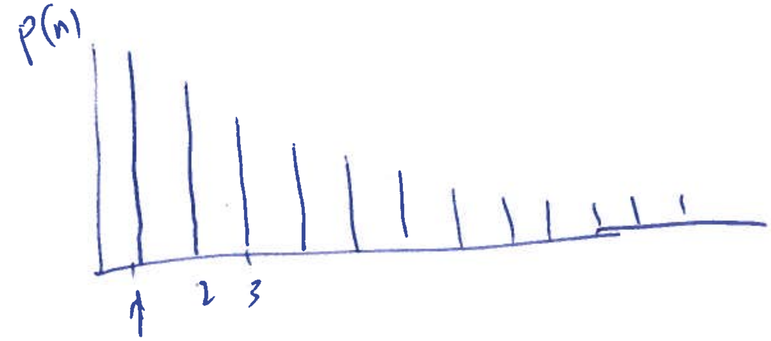
2023-01-23

More R.V. s

Bernoulli (q) : $p(1) = q$ $p(0) = 1-q$

Geometric(q) : $p(n) = q(1-q)^{n-1}$ for $n = 1, 2, \ldots$

<u>note</u>: $\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x}$ for $|x| < 1$



<u>Binomial</u> $\text{Bin}(n, q)$: describes the # of succ. if repeat an experiment n times, each succ with prob q.

<u>pmf</u>: $p(k) = \binom{n}{k} q^k (1-q)^{n-k}$ for $k = 0, 1, \ldots n$

Note: $\sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} = [q + (1-q)]^n = 1^n = 1$

e.g. 5 dice, $X = (\#\text{ of } 6\text{s showing})$

$X \sim \text{Bin}(5, 1/6)$ $\quad$ $P(X=3) = \binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$

Poisson

$$e^{\lambda} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \qquad \text{i.e.} \qquad \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} = 1$$

$Poi(\lambda)$ is a R.V. with $p(n) = e^{-\lambda} \frac{\lambda^n}{n!}$ $\quad n = 0, 1, 2 \ldots$

Shows ~~&~~p as approx. of binomial with n large, q small.

$\lambda = n \cdot q$ fixed.

Consider $X \sim Bin(n, q)$ $\qquad q = \frac{\lambda}{n}$

$$P(X=k) = \binom{n}{k} q^k (1-q)^{n-k} = \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!} \frac{n(n-1)\cdots(n-k+1)}{n^k} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$$

$$P(X=k) \xrightarrow[n\to\infty]{} \frac{\lambda^k}{k!} \cdot 1 \cdot e^{-\lambda} \cdot 1$$

<u>note</u>: $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ $\qquad (1+\varepsilon) \approx e^{\varepsilon}$

so as $n \to \infty$ $\quad X \longrightarrow \text{Poi}(\lambda)$

<u>Note</u>: Poisson approx is robust.

e.g. Birthday collisions: $\binom{n}{2}$ pairs of people.

each pair has same bday with prob $\frac{1}{365}$.

Let X = # of pairs with common bday.

$X \approx \text{Poi}\left(\binom{n}{2} \cdot \frac{1}{365}\right)$

note: X is not exactly binomial, pairs not indep.

For 3 people with same bday:

$$X \approx \mathrm{Poi}\left(\binom{n}{3} \cdot \frac{1}{365^2}\right)$$

$$P(X=0) \approx P(\mathrm{Poi}(\lambda)=0) = e^{-\lambda}$$

so $P(\text{no triplet}) \approx \exp\left(-\binom{n}{3}\frac{1}{365^2}\right)$

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e.g. Repeat exp. 1000 times, each succ. w.p $\frac{1}{500} = q$

$$P(\text{at most 2 succ.}) = \binom{1000}{0} q^0 (1-q)^{1000} + \binom{1000}{1} q (1-q)^{999} + \binom{1000}{2} q^2 (1-q)^{99}$$

$$\approx P(\mathrm{Poi}(2) \le 2) = e^{-\lambda} + e^{-\lambda}\cdot\frac{\lambda}{1} + e^{-\lambda}\frac{\lambda^2}{2}$$

$\lambda =$

$$= e^{-2}\left(1 + 2 + \frac{2^2}{2}\right) = 5e^{-2}$$

Expectation (of discrete RV)

denoted $E(X)$ or $\langle X \rangle$ (in physics)

$$EX = \sum_i x_i \, p(x_i)$$

e.g. $X \sim \text{Bern}(q)$: $EX = q \cdot 1 + (1-q)0 = q$

e.g. $X \sim \text{Geom}(q)$ $EX = \sum_{n=1}^{\infty} n \cdot q (1-q)^{n-1} = 1/q$

(idea : $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$$f' = \sum_n n x^{n-1} = \left(\frac{1}{1-x}\right)^2$$