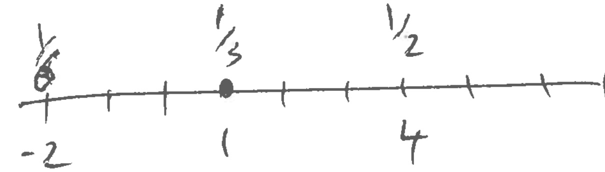
2025-01-25

<u>Recall</u>: pmf of discrete RV: $p(x) = P(X=x)$

$$EX = \sum_i x_i p(x_i)$$

e.g. $p(1) = \frac{1}{3}$ $\quad p(4) = \frac{1}{2}$ $\quad p(-2) = \frac{1}{6}$



centre of mass is the $EX = \frac{1}{6}(-2) + \frac{1}{3}\cdot 1 + \frac{1}{2}\cdot 4$

<u>Geom</u>: $p(n) = q(1-q)^{n-1}$ for $n = 1, 2, \ldots$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$$

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \sum_{n=0}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \ldots$$

For $x = 1-q$: $\quad \frac{1}{q^2} = \sum_{n=0}^{\infty} n(1-q)^{n-1}$ $\qquad \frac{1}{q} = \sum_{n=0}^{\infty} n q (1-q)^{n-1} = \sum n p(n$

Binomial: $p(k) = \binom{n}{k} q^k (1-q)^{n-k}$ for $Bin(n,q)$

Claim: $E\,Bin(n,q) = nq$

$$(1-q+qx)^n = \sum_{k=0}^{n} \binom{n}{k} (qx)^k (1-q)^{n-k} = \sum_{k=0}^{n} x^k \cdot p(k)$$

$$\frac{d}{dx}(\text{same}) = n(1-q+qx)^{n-1} \cdot q$$

$$= \sum_{k=0}^{n} k x^{k-1} p(k)$$

Set $x=1$: $n(1-q+q)^{n-1} q = \sum k\, p(k)$

$$n \cdot q = E\,Bin(n,q)$$

<u>Poi(λ)</u>: $E\,\mathrm{Poi}(\lambda) = \lambda$

<u>Idea</u>: If X takes values in $\mathbb{N}$

$f(x) = \sum p(n) \cdot x^n$ has $f(1) = 1$

$$f'(1) = \sum p(n) \cdot n \cdot 1^{n-1} = EX$$

<u>Poi(λ)</u>: $f(x) = \sum e^{-\lambda} \frac{\lambda^n}{n!} \cdot x^n = e^{\lambda x - \lambda}$

$$f'(x) = \lambda e^{\lambda x - \lambda}$$

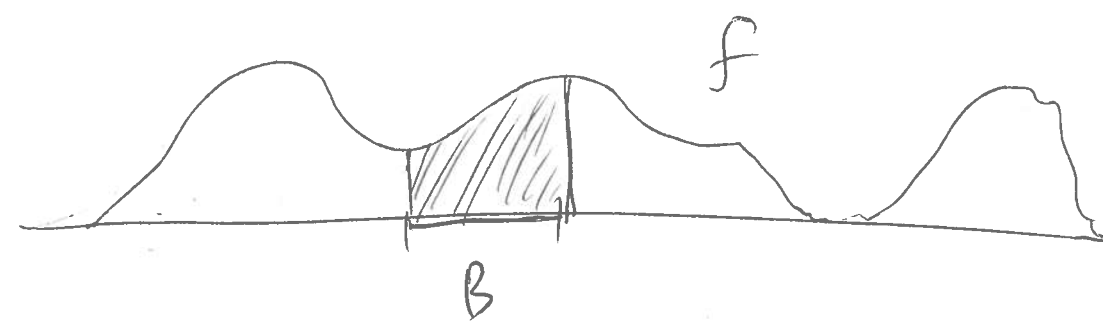
$$f'(1) = \lambda$$

Continuous R.V.s

Defn: X is a conts R.V. if there is a function f (denoted probab. density func, pdf of X) such that

$$P(X \in B) = \int_B f(x)\,dx$$

requires $f \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$



Note: $P(X = a) = 0$ for any a

$$= \int_a^a f(x)\,dx = 0$$

f is not a probab. can have $f(x) > 1$ for some x.

what is $P(X \in (a-\varepsilon, a+\varepsilon))$

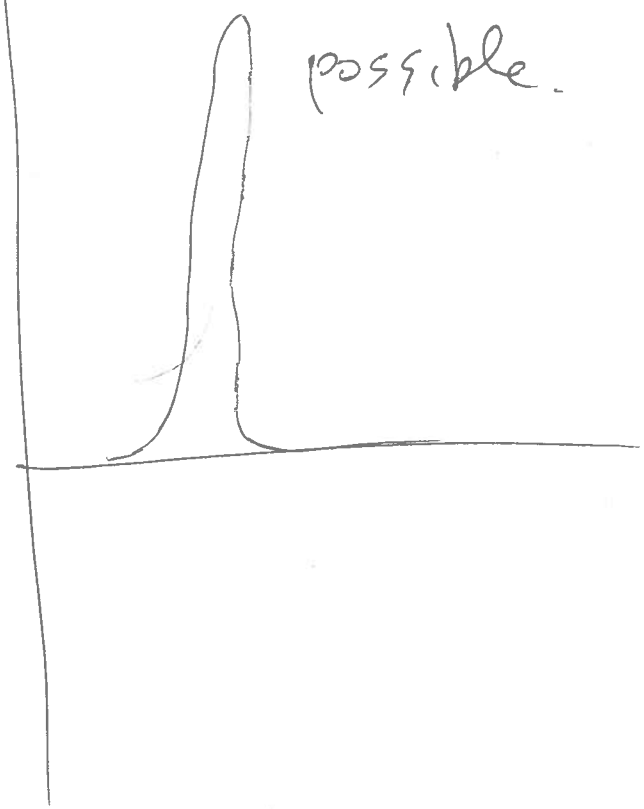
$$\approx 2\varepsilon \cdot f(a)$$

CDF : $F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx$

$$F(a) = P(X < a)$$

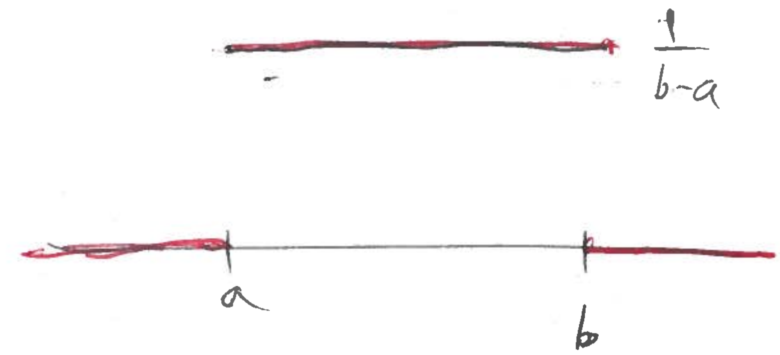
fundamental thm of calculus:

$$F'(a) = f(a)$$



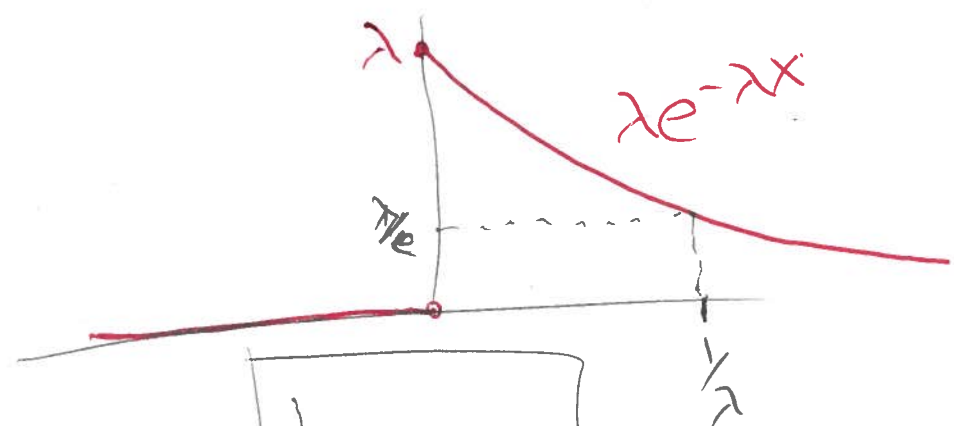
possible.

examples: Uniform in [a,b]



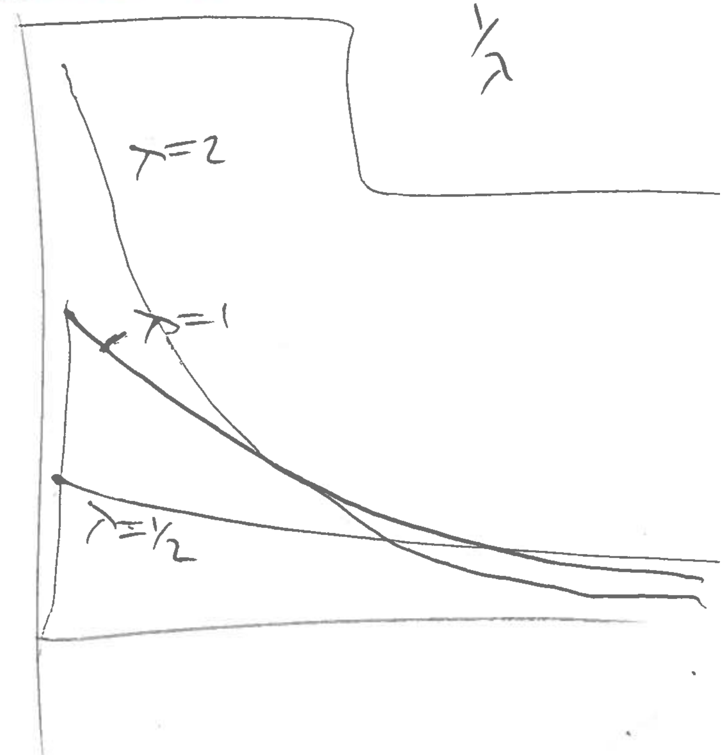
$$f(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{if not.} \end{cases}$$

Exponential: Exp (λ):



$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^{\infty} = 1$$



Often models waiting times for unexpected events

Memoryless Property: Let $X \sim \mathrm{Exp}(\lambda)$

$$P(X \geq s+t \mid X \geq s) = P(X \geq t) \qquad s,t \geq 0$$

$$P(X \geq t) = \int_t^\infty \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_t^\infty = e^{-\lambda t}$$

$$\boxed{P(X \geq t) = e^{-\lambda t}}$$

$$P(X \geq t+s) = e^{-\lambda(t+s)} \quad \text{so} \quad P(X \geq s+t \mid X \geq s) = \frac{P(X \geq s+t)}{P(X \geq s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t}$$

half-life: a s.t. $P(X \geq a) = \frac{1}{2}$

$$e^{-\lambda a} = \frac{1}{2} \quad \text{so} \quad a = \frac{\log 2}{\lambda}$$