2023-01-30

Note:

If a problem asks for $P(N(\mu,\sigma^2) \le t)$ (or similar), Express the sol. in terms of Φ

$\Phi(x) = P(N(0,1) \le x)$

If $X \sim N(\mu,\sigma^2)$ then $X = \mu + \sigma Y$ with $Y = N(0,1)$

$$X \le t \iff \sigma Y + \mu \le t$$
$$\iff Y \le \frac{t-\mu}{\sigma}$$

$$P(X \le t) = \Phi\left(\frac{t-\mu}{\sigma}\right)$$

σ called standard deviation of X

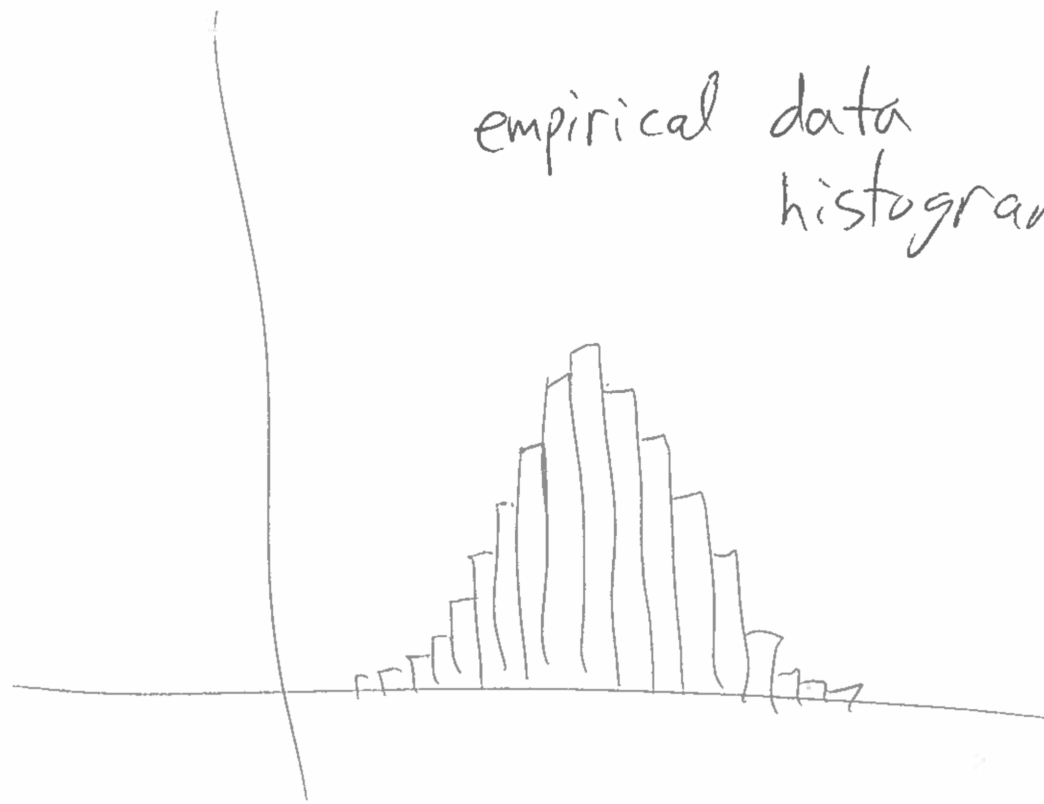
<u>Recall</u> For a cts R.V. X with pdf f

$$EX = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$E\, N(\mu, \sigma^2) = \mu$$

$$E\, Exp(\lambda) = 1/\lambda$$

$$E\, Unif[a,b] = \frac{a+b}{2}$$



empirical data histogram.

If $g : R \to R$ any func. and X is a R.V.

then $g(X)$ is a new R.V.

<u>Thm</u>: $E\,g(x) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx$ for cts X

$E\,g(X) = \sum_i g(x_i) \cdot p(x_i)$ for disc. X

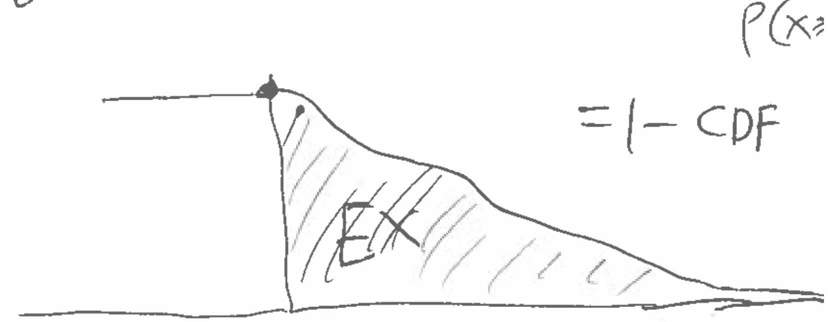
Important special case: $g(X) = aX + b$

$$\int (a\cancel{x}+b) f(x)\,dx = \underbrace{\int a x f(x)\,dx}_{aEX} + \underbrace{\int b f(x)\,dx}_{b} = aEX + b$$

$$\boxed{E(aX+b) = aEX + b}$$

Linearity of E.

<u>Thm</u>: If $X \geq 0$ then $EX = \int_0^\infty P(X \geq t)\,dt$

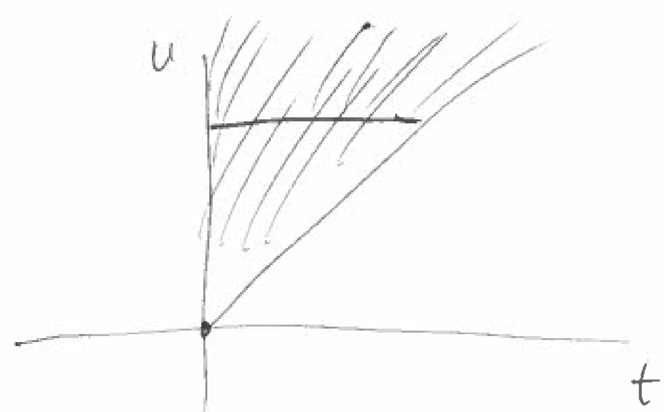


<u>pf</u> (for cts RV)

$$\int_0^\infty P(X \geq t)\,dt = \int_0^\infty \int_t^\infty f(u)\,du\,dt$$

$$= \int_0^\infty \cancel{du} \left(\int_0^u f(u)\,dt \right) du$$

$$= \int_0^\infty du\; u\, f(u) = EX$$



$\square$

<u>note</u>: If X takes values $0, 1, 2, 3 \ldots$

$$EX = \sum_{n=1}^{\infty} P(X \geq n)$$

Pf of $Eg(X) = \int g(x) f(x)\,dx$

If $g \geq 0$ then $g(X) \geq 0$, then

$$Eg(X) = \int_0^\infty P(g(X) \geq t)\,dt = \int_0^\infty \int_{B_t} f(x)\,dx\,dt \qquad B_t = \{x : g(x) \geq t\}$$

$$= \int_{-\infty}^{\infty} \int_0^{g(x)} f(x)\,dt\,dx$$

$$= \int_{-\infty}^{\infty} f(x) \cdot g(x)\,dx \qquad \square$$

Moments: n^{th} moment of X is $E(X^n) = \begin{cases} \int x^n f(x)\,dx \\ \sum x_i^n\, p(x_i) \end{cases}$

note: $E(X^n) \neq (EX)^n$ in general.

Usually 1st moment is denoted by μ.

Variance

<u>def</u>: $\mathrm{Var}(X) = E\left((X - EX)^2\right) = E(X-\mu)^2$ if $\mu = EX$

Often denoted $\sigma^2 = \mathrm{Var}(X)$

σ = standard deviation $= \sqrt{\mathrm{Var}(X)}$

Second formula: $\mathrm{Var}(X) = E(X^2) - (EX)^2$

<u>Proof</u>: $\mathrm{Var}(X) = E\left((X-\mu)^2\right) = E\left(X^2 - 2X\mu + \mu^2\right)$

$$= \int (x^2 - 2x\mu + \mu^2) f(x)\,dx = \int x^2 f(x)\,dx - \int 2\mu x f(x)\,dx + \int \mu^2 f(x)\,dx$$

$$= E(X^2) - 2\mu \underbrace{(EX)}_{=\mu} + \mu^2 = E(X^2) - \mu^2$$

Can find Var of many R.V.s:

$$\mathrm{Var}(\mathrm{Bin}(n,p)) = np(1-p)$$

$$\mathrm{Var}(\mathrm{Pois}(\lambda)) = \lambda$$

$$\mathrm{Var}(\mathrm{Exp}(\lambda)) = 1/\lambda^2$$

$$N(\mu, \sigma^2): \quad \mathrm{Var} = \sigma^2$$