Recall: $Var(X) = E\left((X-\mu)^2\right) = E(X^2) - \mu^2$

where $\mu = EX$

$$Var(aX+b) = a^2 Var(X)$$

Proof: $Var(X+b) = Var(X)$ since $E(X+b) = (EX) + b$

$$Var(aX) = E(aX)^2 - (EaX)^2$$

$$= a^2E(X^2) - a^2(EX)^2 = a^2 Var(X)$$

Joint distributions

X, Y discrete R.V.s have joint p.m.f. $p(x,y) = P(X=x, Y=y)$

The marginal pmf of X is $p_X(x) = P(X=x)$

of Y is $p_Y(y) = P(Y=y)$

$$\boxed{\underline{\text{Claim}}: \quad p_X(x) = \sum_y p(x,y) \qquad p_Y(y) = \sum_x p(x,y)}$$

$y \backslash x$	1	2	3	p_Y
1	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
4	$\frac{1}{6}$	$\frac{1}{3}$	0	$\frac{1}{2}$
p_X	$\frac{10}{24}$	$\frac{11}{24}$	$\frac{3}{24}$	

entries sum to 1.

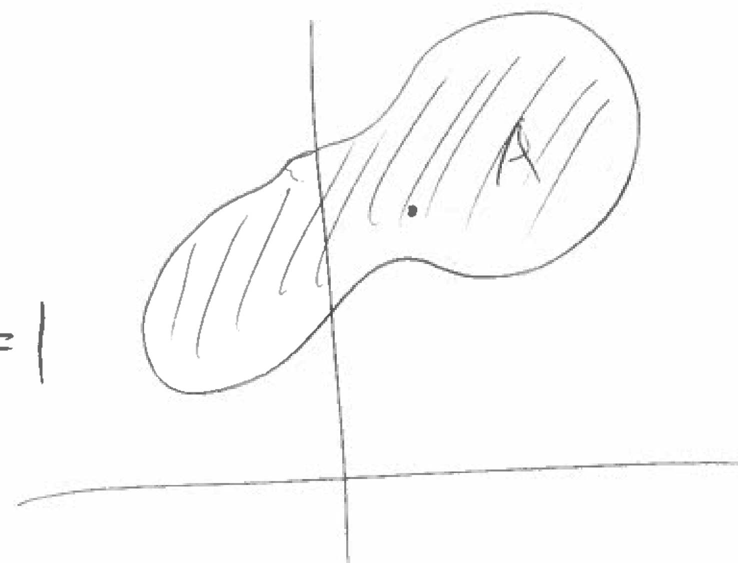
e.g. toss 4 dice. $X = \#$ distinct results
$Y =$ minimal result.

$p(1,3) = P(\{(3,3,3,3)\}) = \frac{1}{6^4}$

$p(4,3) = P(\text{dice show } 3,4,5,6 \text{ in some order}) = \frac{4!}{6^4}$

X, Y are jointly cts R.V.s with pdf $f(x,y)$ if

$$P((X,Y) \in A) = \iint_A f(x,y)\, dx\, dy$$



$f \geq 0$ everywhere, and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

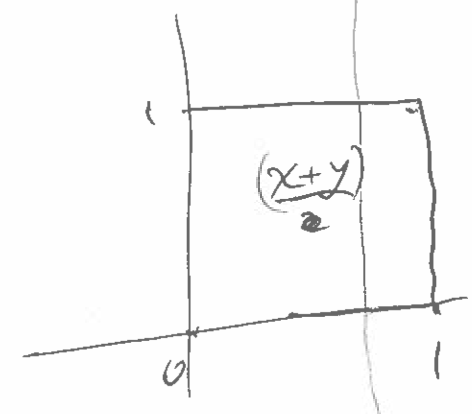
Often A rectangle

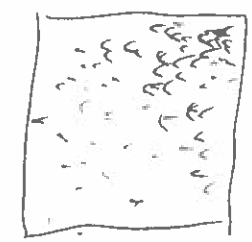
$$P((X,Y) \in A) = \int_{I_1} dx \int_{I_2} dy\, f(x,y)$$

cts marginals: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

e.g. $f(x,y) = \begin{cases} (x+y) & \text{if } 0 \le x, y \le 1 \\ 0 & \text{if not} \end{cases}$



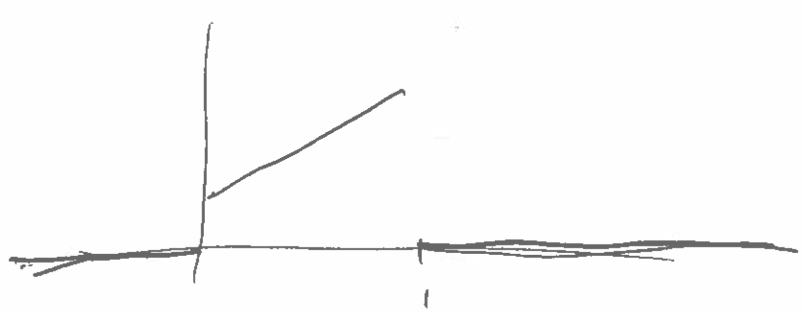


many samples

x-marginal: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$= \int_0^1 f(x,y)\,dy = \int_0^1 (x+y)\,dy$ if $0 \le x \le 1$

$= \frac{x}{1} + \frac{1}{2}$ if $0 \le x \le 1$



Thm $E\,g(X,Y) = \begin{cases} \iint g(x,y)\, f(x,y)\, dx\, dy & X\,Y \text{ cts} \\ \sum_x \sum_y g(x,y)\, p(x,y) & X, Y \text{ discrete} \end{cases}$

for any g

eg. $g(X,Y) = X+Y$: $\boxed{E(X+Y) = EX + EY}$

since $E(X+Y) = \iint (x+y) f(x,y)\, dx\, dy = \iint x f(x,y)\, dx\, dy + \iint y f(x,y)\, dx\, dy$

$$\iint x f(x,y)\, dx\, dy = \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} x f(x,y)\, dy \right)}_{x \cdot f_X(x)} dx = EX$$

second term $= EY$

__Similarly__ : $\sum_{i=1}^{k} EX_i = E\left(\sum_{i=1}^{k} X_i\right)$

Independent R.V.

Defn: X, Y are independent if for any sets A, B,

events $X \in A$ $Y \in B$ are indep: ~~$P(X \in A, Y \in B) =$~~

$$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B).$$

enough to use sets $A = (-\infty, a]$ $B = (-\infty, b]$

equiv. def: for any $a, b \in \mathbb{R}$ $P(X \le a, Y \le b) = P(X \le a)\, P(Y \le b)$.

For disc. R.V. X, Y, indep $\iff p(x,y) = p_X(x) \cdot p_Y(y)$

For cts R.V.s: $\iff f(x,y) = f_X(x) \cdot f_Y(y)$