2023-02-14

$X_1, X_2 \ldots$ are iid $\mathrm{Exp}(\lambda)$

$S_n = \sum_{i=1}^{n} X_i$ = time of nth event.

Seen: $S_n \approx n/\lambda$: seen that the char. func.

of $\frac{1}{n} S_n$ $\cancel{\xrightarrow[n\to\infty]{}}$ conv. as $n \to \infty$ to $e^{it/\lambda}$.

This is the char func. of const. $\frac{1}{\lambda}$.

Convergence of distributions

def: Let $F_1, F_2 \ldots$ are CDF of some R.V.s $X_1, X_2 \ldots$

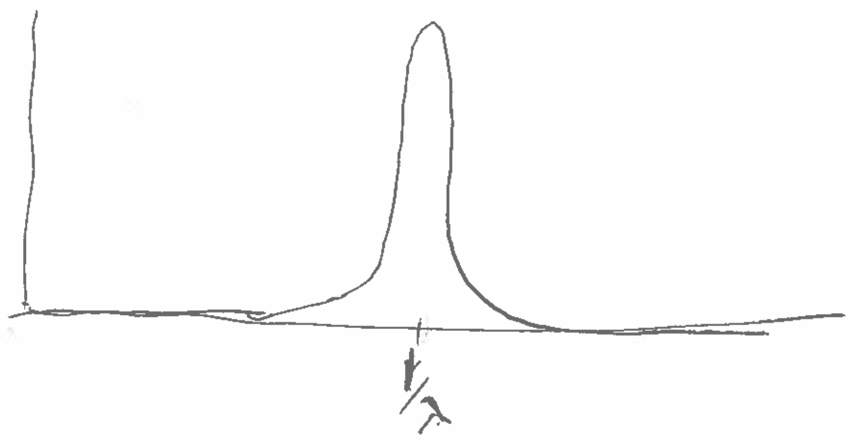
We write $F_n \Rightarrow F$ if

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \text{whenever } F \text{ is cts. at } x.$$

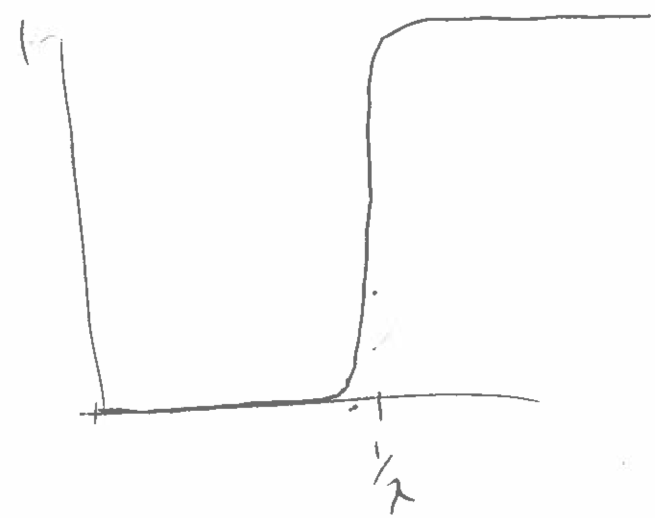
e.g. $F_n = $ CDF of $\frac{1}{n}S_n$ then we will see:

$$F_n(x) \xrightarrow[n\to\infty]{} \begin{cases} 0 & x < 1/\lambda \\ 1 & x > 1/\lambda \end{cases}$$

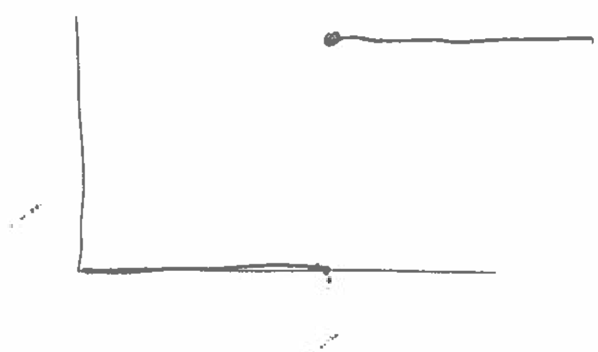
pdf of S_n/n:



CDF

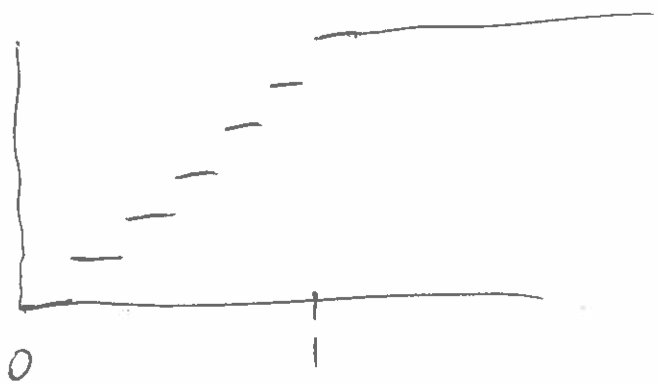


CDF of const. $1/\lambda$

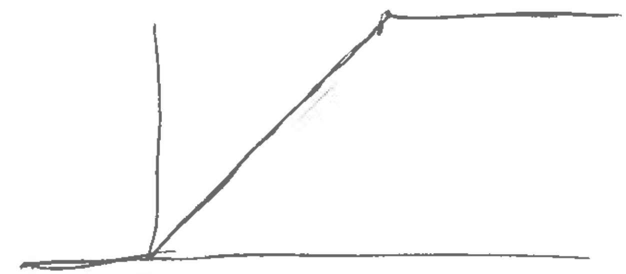


e.g. X_n is unif on $\{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$

CDF of X_n



$\xrightarrow{n \to \infty}$



CDF of Unif$[0,1]$

If R.V.s X_n with CDFs F_n have $F_n \Rightarrow F$

we say X_n converge in distribution.

$X_n \xrightarrow{D} X$ if X has CDF F.

i.e. $F_{X_n}(x) \longrightarrow F_X(x)$ wherever F_X is cts.

__Thm__ (continuity thm) If RVs X_n have CDF F_n and char. funcs ϕ_n,

(a) If $F_n \Rightarrow F$ ~~the~~ and F is CDF of X then $\phi_n \longrightarrow \phi$: char func of X.

(b) If $\phi_n(t) \longrightarrow \phi(t)$ for all t and ϕ is cts at 0 then ϕ is the char. func. of X and $X_n \xrightarrow{D} X$.

e.g. $S_n = \sum_{i=1}^{n} X_i$ $\quad X_n$ iid $\text{Exp}(\lambda)$.

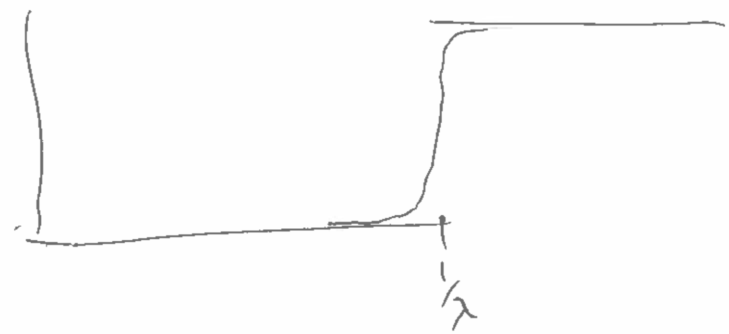
$Y_n = \frac{1}{n} \cdot S_n$

Seen! $\phi_{Y_n}(t) = \phi_{S_n}\left(\frac{t}{n}\right) = \left(1 - \frac{it}{\lambda n}\right)^{-n}$

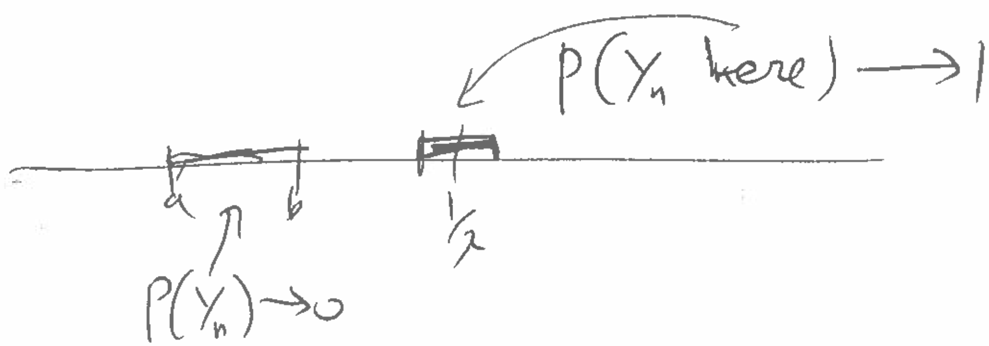
$\xrightarrow[n\to\infty]{} e^{it/\lambda}$

By continuity thm, $Y_n \xrightarrow{D} \frac{1}{\lambda}$

CDF of $Y_n \approx \text{CDF}(\frac{1}{\lambda})$



so $P(a<Y_n<b) \longrightarrow P(a<\frac{1}{\lambda}<b)$



i.e. the average of $X_1 \text{---} X_n$ is $\approx \frac{1}{\lambda}$ with high probab.

Weak Law of Large Numbers (LLN)

Let $X_1, X_2, \ldots$ be iid.

Let $\mu = EX_i$ be finite.

Let $S_n = \sum_{i=1}^{n} X_i$. Then $\frac{1}{n} S_n \xrightarrow{D} \mu$

Pf By continuity thm, suffice to show that

$$\phi_{\frac{1}{n}S_n}(t) \xrightarrow[n\to\infty]{} e^{it\mu} \quad \text{for all } t.$$

$$\phi_{\frac{1}{n}S_n}(t) = \phi_{S_n}\left(\frac{t}{n}\right) = \left(\phi_X\left(\frac{t}{n}\right)\right)^n \qquad \phi_X = \phi_{X_i} \text{ for all } i$$

Taylor: $\phi_X(s) = 1 + s\,\phi_X'(0) + o(s^2)$

$$= 1 + i(EX)s + o(s^2)$$

$$\phi_X\left(\frac{t}{n}\right) = 1 + \frac{it}{n}\mu + o\left(\left(\frac{t}{n}\right)^2\right)$$

$$\phi_X\left(\frac{t}{n}\right)^n = \left(1 + \frac{it}{n}\mu + o\left(\frac{t^2}{n^2}\right)\right)^n \xrightarrow[n\to\infty]{} e^{it\mu} \qquad \square$$

__note:__ continuity thm needs the assump that ϕ is cts at 0.

If $X_n \sim \text{unif}[-n, n]$ then $\phi_n(t) = \begin{cases} \frac{\sin(nt)}{nt} & t \neq 0 \\ 1 & t = 0 \end{cases}$

$$\phi_n(t) \xrightarrow[n\to\infty]{} \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$

This is not the char func. of any R.V.

X_n do not converge.

__Strong LLN:__ $\frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow \mu$ with probab. 1

__Central limit theorem:__

how close is $\frac{1}{n}S_n$ to μ?

$$\frac{\frac{1}{n}S_n - \mu}{\sqrt{n}\cdot\sigma} \xrightarrow{D} N(0,1)$$

std dev of X