Conditional Expectation

Recall: cond. prob $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

<u>Defn</u> The conditional pmf of X given $Y=y$ is

$$P_{X|Y}(x|y) = P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{p(x,y)}{P_Y(y)}$$

The cond expectation of X given $Y=y$ is

$$E[X \mid Y=y] = \sum_x x P_{X|Y}(x|y)$$

e.g.

$y \backslash x$	0	1	2
1	1/8	0	1/8
2	1/8	1/4	0
3	0	1/8	1/4

$$P_{X|Y}(0|2) = \frac{p(0,2)}{P_Y(2)} = \frac{1/8}{1/8 + 1/4} = \frac{1}{3}$$

$$P_{X|Y}(1|2) = \frac{1/4}{3/8} = \frac{2}{3}$$

$$P_{X|Y}(2|2) = 0$$

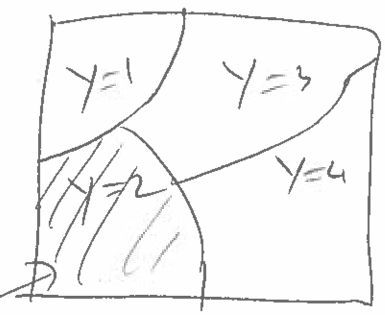
$$E[X|Y=2] = 0 \cdot p_{X|Y}(0|2) + 1 \cdot p_{X|Y}(1|2) + 2\, p_{X|Y}(2|2)$$

$$= 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} + 2 \cdot 0 = \frac{2}{3}.$$

<u>note</u>: for any y, $\sum_x p_{X|Y}(x|y) = 1$

<u>Note</u>: $E[X|Y]$ is a R.V. that takes the value

$E[X|Y=y]$ when $Y=y$.



$E[X|Y] = E[X|Y=2]$

<u>e.g.</u> X = # spades Y = # Hearts in 13 card hand.

$EX = EY = \frac{13}{4}$.

$E[X|Y=0] = \frac{13}{3}$ $\quad E[X|Y=2] = \frac{11}{3}$

In general $E[X|Y] = \frac{13-Y}{3}$

<u>Thm</u>: $EX = \sum_y p_Y(y) \cdot E[X|Y=y] = E[E[X|Y]]$

In last example: $EX = \sum_y p_Y(y) \cdot \frac{13-y}{3} \overset{?}{=}$

Indeed $\sum p(y) \cdot \frac{13-y}{3} = \sum_y p(y) \cdot \frac{13}{3} - \sum_y p(y) \cdot \frac{y}{3}$

$$= \frac{13}{3} - \frac{1}{3} EY = \frac{13}{3} - \frac{1}{3} \cdot \frac{13}{4} = \frac{13}{4}$$

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e.g. Game: draw a card.

$\left.\begin{array}{ll} \text{J, Q, K}: & \text{win } 10 \\ 2 \text{---} 10 & \text{win } 1 \end{array}\right\}$ repeat.

Ace : game ends

cards shuffled each time.

X = total prize.

Qn : Find EX

Sol. Let Y be the amount won in first round.
Let Z be the rest. $X = Y + Z$.

w.p. $\frac{1}{13}$ $X = Y = Z = 0$

w.p. $\frac{9}{13}$ $Y = 1$ $\quad E[X \mid Y=1] = 1 + E[X]$

w.p. $\frac{3}{13}$ $Y = 10$ $\quad E[X \mid Y=10] = 10 + E[X]$

$$\left.\begin{aligned} E[X \mid Y=1] &= 1 + E[X] \\ E[X \mid Y=10] &= 10 + E[X] \end{aligned}\right\} EZ = EX$$

$$E[X] = \sum_y P(y) E[X \mid Y=y] = \frac{1}{13} \cdot 0 + \frac{9}{13}(1 + EX) + \frac{3}{13}(10 + EX)$$

$$EX = \frac{39}{13} + \frac{12}{13} EX$$

$$EX = 39$$

<u>Proof of thm</u>: $EX = \sum x\, p_X(x) = \sum_x x \sum_y p(x,y)$

$$EX = \sum_y \sum_x x\, p(x,y)$$

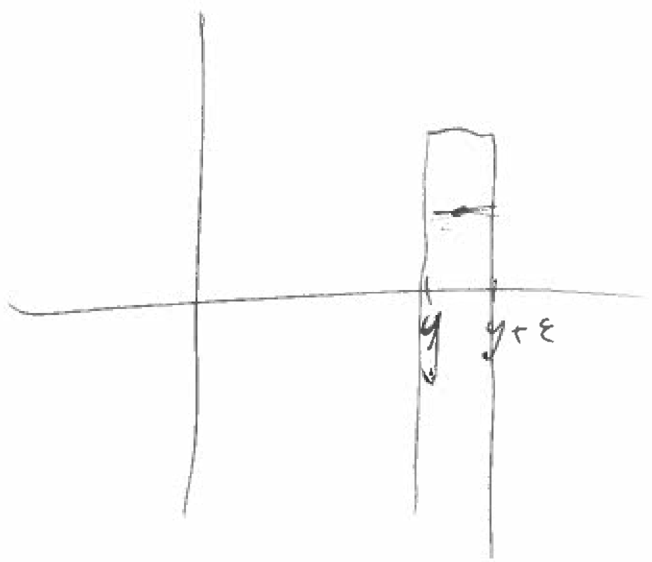
$$= \sum_y \sum_x x\, p_Y(y) \cdot p_{X|Y}(x|y) = \sum_y p_Y(y) \cdot \underbrace{\sum_x x\, p_{X|Y}(x|y)}_{E[X|Y=y]}$$

$\square$

<u>Contin. case</u>: want pdf $f(x|y)$ = density of x given $Y=y$.

But $P(Y=y)=0$ so can't condition on $Y=y$.

$$P(X \le x \mid y \le Y \le y+\varepsilon) = \frac{\int_{-\infty}^{x} \int_{y}^{y+\varepsilon} f(u,v)\, du\, dv}{\int_{y}^{y+\varepsilon} f_Y(v)\, dv}$$



Can take a limit as $\varepsilon \to 0$

$$P(X \le x \mid y \le Y \le y+\varepsilon) \approx \frac{\varepsilon \int_{-\infty}^{x} f(u,y)\,du}{\varepsilon \cdot f_Y(y)}$$

limit is $\displaystyle\int_{-\infty}^{x} \frac{f(u,y)}{f_Y(y)}\,du$

conditional pdf of X given $Y=y$.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$E[X|Y=y] = \int x \cdot f_{X|Y}(x|y)\,dx$$

e.g. X, Y have joint pdf $x+y$ on $[0,1]\times[0,1]$

$$f_Y(y) = \int_0^1 x+y \, dx = y + \frac{1}{2} \quad \text{on } [0,1]$$

$$f_{X|Y}(x|y) = \frac{x+y}{y+1/2}$$

$$E[X|Y=y] = \int_0^1 x \cdot \frac{x+y}{y+1/2} \, dx = \frac{1}{3}\frac{1}{y+1/2} + \frac{1}{2}\frac{y}{y+1/2}$$