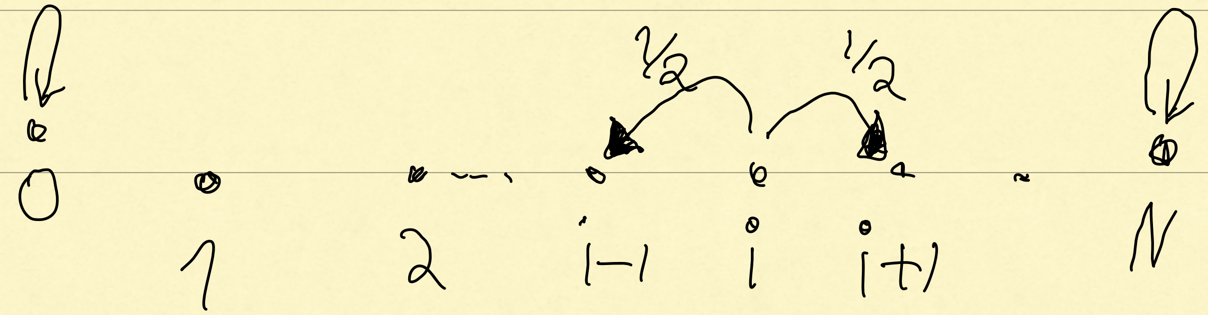
Markov Chains:

Examples: Gambler's ruin:
"Simple Random Walk on $\{0,1,\dots,N\}$ with absortion at the boundary $\{0,N\}$."

Player starts with n \$,
Bank with m
Total $N=n+m$.

At each step (time unit) bet 1\$ on a fair coin flip, until the player has 0 or N \$

(whichever occurs first)



$A :=$ Player reaches N.

Let $q_n := P(A \mid X_0 = n)$

Problem: Find q_n.

Sol:

$$q_n := P\left(A \,\middle|\, \begin{matrix} X_0 = n \text{ and} \\ \text{win } 1^{st} \\ \text{coin flip} \end{matrix}\right) P\left(\begin{matrix}\text{win } 1^{st} \\ \text{coin flip}\end{matrix}\right)$$

$$+ P\left(A \,\middle|\, \begin{array}{l} X_0 = n \text{ and} \\ \text{lose 1st} \\ \text{coin flip} \end{array}\right) P\left(\begin{array}{l} \text{lose 1st} \\ \text{coin flip} \end{array}\right)$$

$$= \frac{1}{2} q_{n+1} + \frac{1}{2} q_{n-1}$$

Multiplying by 2 and rearanging, we get that

$\Delta_n = \Delta_{n+1}$ where

$\Delta_j = q_j - q_{j-1}$

This means that $(q_n)_{n=0}^{N}$

is an arithmetic progression.

We get that

$\Delta_1 = \Delta_2$, $\Delta_2 = \Delta_3$, $\Delta_3 = \Delta_4$,

$\ldots$, $\Delta_{N-2} = \Delta_{N-1}$ and $\Delta_{N-1} = \Delta_N$

Denote the common value of $\Delta_1, \Delta_2, \ldots, \Delta_N$ by Δ.

Then $q_0 = 0$

$$q_1 = q_0 + \Delta_1 = q_0 + \Delta = \Delta$$

$$q_2 = q_1 + \Delta_2 = q_1 + \Delta = 2\Delta$$

By induction $q_i = i\Delta$. Indeed

$$q_{i+1} = q_i + \Delta_{i+1} = i\Delta + \Delta = (i+1)\Delta$$

which establishes the induction step. Finally

$$1 = q_N = N\Delta \Rightarrow \Delta = 1/N$$

$$\Rightarrow q_n = n\Delta = n/N.$$

E.g. if $n = 100$, $N = 1000$ then

$$q_{100} = 1/10.$$

Alternative solution:

Consider all sequences $(u_n)_{n=0}^{N}$ satisfying the system of linear equations

$$\boxed{(*) \quad u_n = \frac{1}{2} u_{n-1} + \frac{1}{2} u_{n+1} \quad \text{for all } n=1,2,\dots,N-1}$$

This is a vector subspace of the space of all sequences of real numbers labeled by $\{0,1,\dots,N\}$ (which is essentially $\mathbb{R}^{N+1}$).

It has dimension 2 since the values of u_0 and u_1 determine the values of u_n for all $n=2,3,\dots,N$ via $(*)$

Exercise: see you understand why this is indeed the case.

Look for a solution of the form $u_n = x^n$ $(x \neq 0)$

Plugging this in ⊛ gives

$$x^n = \frac{1}{2} x^{n-1} + \frac{1}{2} x^{n+1}$$

$$\Longleftrightarrow \quad x = \frac{1}{2} + \frac{1}{2} x^2$$

The solution is $x_1 = 1 = x_2$.

Since 1 is a double root the general solution to ⊛ is of the form

$u_n = a \cdot 1^n + b \cdot 1^n n$ for some

$a, b \in \mathbb{R}$.

One can check this is a solution ✓ for all $a, b \in \mathbb{R}$

and the collection of such solutions is a vector space of dim 2.

$$0 = q_0 = a \cdot 1^0 + b \cdot 1^0 \cdot 0 = a \Rightarrow a = 0$$

$$1 = q_N = b \cdot 1^N \cdot N \Rightarrow b = 1/N$$

$$\Rightarrow q_n = n/N.$$

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② What if bets are biased?

"Biased Random Walk on $\{0, 1, \dots, N\}$ with absortion at the boundary $\{0, N\}$."

E.g. Roulette: 18 red, 18 black, 2 green.

Bet 1 \$ each time unit on red.

Win w.p. (with probability) $p = \frac{18}{38}$,

lose w.p. $1-p = \frac{20}{38}$.

By the same logic

$$q_n := P\left(A \,\middle|\, \begin{array}{l} X_0 = n \text{ and} \\ \text{win } 1^{st} \text{ bet} \end{array}\right) P\left(\begin{array}{c}\text{win } 1^{st} \\ \text{bet}\end{array}\right)$$

$$+ P\left(A \,\middle|\, \begin{array}{l} X_0 = n \text{ and} \\ \text{lose } 1^{st} \\ \text{bet} \end{array}\right) P\left(\begin{array}{c}\text{lose } 1^{st} \\ \text{bet}\end{array}\right)$$

$$\Longrightarrow \boxed{(*) \quad q_n = p\, q_{n+1} + (1-p)\, q_{n-1} \quad \text{for all } n = 1, 2, \ldots, N-1.}$$

Again the space of all solutions

to the system of linear equations:

$$\boxed{(**) \quad u_n = p\,u_{n+1} + (1-p)\,u_{n-1} \qquad n = 1, 2, \dots, N-1}$$

is a vector space of dim 2.

Look for a solution of the form $u_n = x^n$ $(x \neq 0)$

Plugging this in $(*)$ gives

$$x^n = p\,x^{n+1} + (1-p)\,x^{n-1}$$

$$\iff \quad x = 1-p + p\,x^2$$

The two solutions are

$$x_1 = 1 \quad \text{and} \quad x_2 = \alpha := \frac{1-p}{p}$$

Hence for any $a, b \in \mathbb{R}$

$u_n = a + b\alpha^n \qquad n = 0, 1, \ldots, N$

is a solution to (**).

This is a collection of solutions of dimension 2 and hence any solution to (**) is of this form.

Using the boundary conditions $q_0 = 0$ and $q_N = 1$ yields

$$0 = a + b$$

$$1 = a + b\alpha^N$$

$$\Rightarrow \quad a = \frac{1}{1-\alpha^N} = -b$$

$$\Rightarrow \quad q_n = \frac{1-\alpha^n}{1-\alpha^N}$$

Since $\alpha > 1$ we get that

q_{N-k} is exponentially small in k

$$\boxed{q_{N-k} \leq \alpha^{-k}.}$$

Hence q_{N-10} is small.

Simple Random Walk (SRW) on $\mathbb{Z}^d$

Defn: Let

$$\mathbb{Z}^d := \left\{ (z_1, \ldots, z_d) : \begin{array}{l} z_i \text{ is an} \\ \text{integer} \\ i = 1, 2, \ldots, d \end{array} \right\}$$

be the collection of all d-tuples of integers.

Let

$$e_j := (0, \ldots, 0, \underset{\substack{\uparrow \\ j\text{-th coordinate}}}{1}, 0, \ldots, 0)$$

be the unit vector in

direction j.

Let $(\vec{X_n})_{n=1}^{\infty}$ be i.i.d.

satisfying for $i = 1, 2, \ldots, d$

$$P(\vec{X_n} = e_i) = \frac{1}{2d} = P(\vec{X_n} = -e_i).$$

We define $\vec{S_0} = \vec{0}$ and

$\vec{S_n}$ = the position of the walk at time n

$$:= \sum_{k=1}^{n} \vec{X_k}.$$

Then $\vec{X}_k$ is the increment of the walk at time k.

Verbal description: At each time n the walk picks a random direction $i_n \sim \mathrm{Unif}\{1, \dots, d\}$ and then w.p. $\frac{1}{2}$ it moves by $+1$ in this direction ($\vec{S}_{n+1} = \vec{S}_n + e_{i_n}$) and w.p. $\frac{1}{2}$ it moves by -1 in this direction ($\vec{S}_{n+1} = \vec{S}_n - e_{i_n}$).

Let $u := P\left(\begin{array}{l}\text{The walk eventually} \\ \text{returns to } \vec{0}\end{array}\right)$

$$= P(\exists\, n > 0 \text{ s.t. } \vec{S_n} = \vec{0})$$

Problem: Determine whether $u = 1$.

Note $u \geq \frac{1}{2d} = P(\vec{S_2} = 0) = \frac{2d}{(2d)^2}$

Observe that if $u = 1$ then $\vec{S}$ returns to $\vec{0}$ infinitely often.

Since after each return to $\vec{o}$ the walk starts afresh, and thus returns once again w.p. u.

Let $N := \{n \geq 0 : \vec{S_n} = \vec{o}\}$

$= \#$ visits to $\vec{o}$

$= 1 + \#$ returns to $\vec{o}$

The above reasoning also yields:

Claim: If $u < 1$ then $N \sim \text{Geom}(1-u)$

i.e. $P(N=k)=u^{k-1}(1-u)$ $k=1,2,\dots$

Pf: Time 0 contributes $+1$ to N. After each visit to $\vec{0}$ the walk starts afresh and thus has probability u of returning and $1-u$ of not returning to $\vec{0}$.

Hence for all $k \geq 1$

$$P(N \geq k)=u^{k-1}.$$

$$\Rightarrow \quad P(N=k)=P(N\geq k)-P[N \geq k+1]$$

$$= u^{k}-u^{k+1}=u^{k}(1-u) \quad \blacksquare$$

Denote

$$m := EN = E \sum_{n=0}^{\infty} 1\{\vec{S_n} = \vec{0}\}$$

$$= \sum_{n=0}^{\infty} E 1\{\vec{S_n} = \vec{0}\}$$

$$= \sum_{n=0}^{\infty} P(\vec{S_n} = \vec{0}).$$

By the above discussion

if $u=1$ then $m=\infty$ and

if $u<1$ then $m = \frac{1}{1-u} < \infty$.

$$\Longrightarrow \boxed{u = 1 - \frac{1}{m}}.$$

Hence $u=1$ if and only if $m=\infty$.