M.C.s (chapter 4)

e.g. Gamblers ruin:

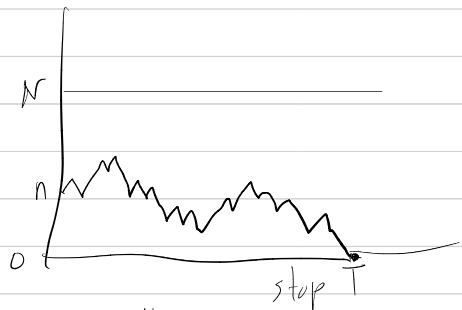
Player starts with $n$, Bank with $m$

total $N = n+m$.

each time bet 1 on fair coin until

player has 0 or $N$. [Simple RW, absorbing bdry]

Q: Find $P(\underbrace{\text{Player reaches}}_{A} N)$




Sol: Let $q_n = P(A \mid X_0 = n)$

$q_0 = 0 \quad q_N = 1$

$q_n = P(A \mid \text{win } 1^{st}) \cdot P(\text{win } 1^{st}) + P(A \mid \text{lose } 1^{st})\, P(\text{lose } 1^{st})$

$q_n = \frac{1}{2} \cdot q_{n+1} + \frac{1}{2} q_{n-1}$

To solve: $2q_n = q_{n-1} + q_{n+1}$

$q_{n+1} - q_n = q_n - q_{n-1}$

so Arith. prog.

$q_n = an + b$ , since $q_0 = 0$ $q_N = 1$

get $q_n = \frac{n}{N}$

e.g. $n = 100$ $N = 1000$ $\Rightarrow q = \frac{1}{10}$.

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what if bets are biased (biased RW)

e.g. Roulette: 18 red 18 black 2 green

bet on red: win w.p. $p = \frac{18}{38}$ lose $1-p = \frac{20}{38}$

$$q_n = p\, q_{n+1} + (1-p) q_{n-1} \qquad (*)$$

sol.: guess sol. of type $q_n = x^n$

$$x^n = p x^{n+1} + (1-p) x^{n-1}$$

$$x = p x^2 + 1 - p \quad \Rightarrow \quad x = 1 \text{ or } x = \alpha = \frac{1-p}{p}$$

gen. sol of $(*)$ is

$$q_n = a \cdot 1^n + b\alpha$$

bdry cond gives $a, b$: $a 1^0 + b\alpha^0 = 0$

$a 1^N + b\alpha^N = 1$

so $\quad a = -\frac{1}{\alpha^N - 1} \qquad b = \frac{1}{\alpha^N - 1}$

$$q_n = \frac{\alpha^n - 1}{\alpha^N - 1}$$


$q_{990} \approx 0.35$

(unlikely to ever reach $n+10$)

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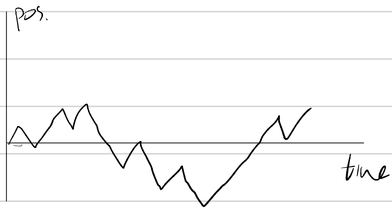
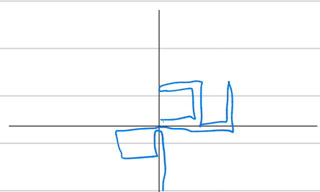
SRW on $\mathbb{Z}^d$ : $\qquad$ (def $\mathbb{Z}^d$ = d-tuples of int)

d=1




d=2



One not in image.

unit vectors $e_i = (0 \ldots 1 \ldots 0) \qquad i = 1 \ldots d$

steps $\vec{X}_n$ iid $\qquad P(\vec{X}_n = e_i) = \frac{1}{2d}$

This is first time we use vector R.V.s

stick to vector notation

$\vec{S}_n = \sum_{i=1}^{n} \vec{X}_i \quad$ is pos. at time $n$

$$\vec{S}_0 = \vec{0}$$

Qn: find $u = P(\text{return to } \vec{0})$
$= P(\exists n > 0 \text{ s.t. } \vec{S}_n = \vec{0})$. (walk never ends)

Note ⊗ $u \geq \frac{1}{2d} = P(\vec{S}_2 = 0) = \frac{2d}{(2d)^2}$

⊗ If $u = 1$ then $\vec{S}$ returns to $\vec{0}$ $\infty$ often since after each return it starts afresh.

⊗ Let $N = \#\{\text{visits to } \vec{0}\} = \#\{n \geq 0 : \vec{S}_n = \vec{0}\}$
inc. 0

Let $m = EN$.

If $u < 1$, $P(N = K) = u^{K}(1-u)$ ⟵ no return after.
0 + return K-1 times

So $N \sim \text{Geom}(1-u)$ $m = EN = \frac{1}{1-u}$ $u = 1 - 1/m$

* $u = 1 - \frac{1}{m}$ also if $u = 1$ since $m = \infty$.

Thm: $u = 1 \iff m = \infty$

$$I_n = \begin{cases} 1 & \vec{S}_n = \vec{0} \\ 0 & \vec{S}_n \neq \vec{0} \end{cases} \quad \text{so} \quad N = \sum_{n=0}^{\infty} I_n$$

$$m = EN = \sum_n E I_n = \sum_{n=0}^{\infty} P(\vec{S}_n = \vec{0})$$

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$d=1$ calc. for biased walk, $P(X_n = 1) = p$
$P(X_n = -1) = 1-p$

$$P(S_n = 0) = \begin{cases} 0 & n \text{ odd} \\ \binom{2k}{k} p^k (1-p)^k & n = 2k \text{ even} \end{cases}$$

Stirling: $$\binom{2k}{k} \sim \frac{\sqrt{2\pi\, 2k}\left(\frac{2k}{e}\right)^{2k}}{\left(\sqrt{2\pi k}\left(\frac{k}{e}\right)^k\right)^2} \sim \frac{4^k}{\sqrt{\pi k}}$$

so $$P(S_{2k} = 0) \sim \frac{1}{\sqrt{\pi k}} \left(4p(1-p)\right)^k$$

If $p \neq 1/2$ then $4p(1-p) < 1$ and $\sum P(S_{2k} = 0) < \infty$.

so $m < \infty$ and $u = 1 - \frac{1}{m} < 1$

If $p = 1/2$ then $\sum P(S_{2k} = 0) = \infty$ by integral test.

$$\int \frac{1}{\sqrt{\pi x}}\, dx = \infty$$

note: LLN also implies transience for $p \neq 1/2$.

$d=2$ (symmetric RW only)

$$P(\vec{S}_n=\vec{0}) = \begin{cases} 0 & n \text{ odd} \\ \left(\binom{2k}{k}\frac{1}{4^k}\right)^2 & n=2k \text{ even} \end{cases}$$

as if coord. are indep. though they are not!

Pf: $X_n, Y_n$ not indep but $\underbrace{X_n+Y_n}_{U_n}$ and $\underbrace{X_n-Y_n}_{V_n}$ are indep.
since $U_n$ and $V_n$ change by $\pm 1$ each step indep. $\square$

note: seen in HW that for $N(0,1)$ $X,Y$ $X\pm Y$ are also indep. This is special for gaussians.

so $P(\vec{S}_{2k}=\vec{0}) \sim \left(\frac{4^k}{\sqrt{\pi k}}\frac{1}{4^k}\right)^2 \sim \frac{1}{\pi k}$

$$m = EN = \sum P(\vec{S}_{2k}=\vec{0}) = \infty$$

<u>Thm</u>: SRW on $\mathbb{Z}^d$ is recurrent for $d\le 2$ and transient for $d>2$

$d=1$ and $d=2$ case done.

$d>2$ case:

<u>claim</u>: $P(\vec{S}_{2n}=\vec{0}) \sim Cn^{-d/2}$

(as if coords are indep. but const. is not the same)

we won't prove this, but one heuristic idea: whp. each coord. moves $\sim \frac{2n}{d}$ times (LLN).

If each coord. moves even # of times (p $= 2^{1-d}$ if asked) then

$$P(\vec{S}_{2n}=\vec{0}) \approx c\left(\frac{1}{\sqrt{\pi n/d}}\right)^d = Cn^{-d/2}.$$

Thus $\sum P(\vec{S}_{2n}=\vec{0}) < \infty$ if $d>2$.

Pf using char. func.:

For vector $\vec{X}$ in d-dim, the char. func. $\varphi$ for $\vec{t} \in \mathbb{R}^d$ is

$$\varphi(\vec{t}) = E\, e^{i\langle \vec{X}, \vec{t}\rangle} \qquad \langle \vec{X}, \vec{t}\rangle = \sum_i x_i t_i$$

Inverse fourier transform special case:

For discrete R.V. $\vec{X}$ with integer coord.

$$\mathbb{P}(\vec{X} = \vec{0}) = (2\pi)^{-d} \int_B \varphi(\vec{t})\, d\vec{t}$$

where $B = [-\pi, \pi]^d$ is d-dim box.

For us, $\vec{X} = \pm e_i$ w.p. $\frac{1}{2d}$ for all $i$

so $\varphi(\vec{t}) = \frac{1}{2d} \sum_j e^{it_j} + e^{-it_j}$

$$\boxed{\varphi(\vec{t}) = \frac{1}{d} \sum_j \cos(t_j)}$$

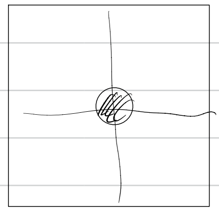
$$\varphi_{\vec{S}_n}(\vec{t}) = \left[\varphi_X(\vec{t})\right]^n$$

so $P(\vec{S}_n = \vec{0}) = (2\pi)^{-d} \int_B \varphi(\vec{t})^n \, d\vec{t}$

$$m \doteq \sum_n P(\vec{S}_n = \vec{0}) = (2\pi)^{-d} \int_B \sum_n \varphi(\vec{t})^n \, d\vec{t}$$

$$= (2\pi)^{-d} \int_B \frac{1}{1 - \varphi(\vec{t})} \, d\vec{t}$$

the integrand is bdd in the box except near $\vec{0}$. (only place where $\varphi = 1$).



split $\int_B = \int_{Ball} + \int_{B \setminus Ball}$

for small ball near 0.

outside ball, $\frac{1}{1-\varphi}$ bdd so 2nd $\int < \infty$.

near 0:

$$\varphi(\vec{t}) = \frac{1}{d}\sum \cos(t_j) \approx \frac{1}{d}\sum 1 - \frac{1}{2}t_j^2 = 1 - \frac{1}{2d}\sum t_j^2$$

$$= 1 - \frac{1}{2d}\|t\|_2^2.$$

Then $\int_{Ball} \frac{1}{1-\varphi} \approx \int_{Ball} \frac{2d}{\|t\|_2^2} = \int_0^{\varepsilon} \frac{2d}{r^2} r^{d-1} dr$

pola coord Jacobian

$$\int_0^{\varepsilon} 2d\, r^{d-3} = \begin{cases} \infty & \text{if } 3-d \geq 1 \quad \text{i.e. } d \leq 2 \\ < \infty & \text{if } 3-d < 1 \quad \text{i.e. } d > 2. \end{cases}$$

Thus if $d > 2$ $m = \int \frac{1}{1-\varphi} < \infty$ and so SRW is transient.

<u>Note</u>: This can be computed. e.g. $d=3$, $m = 1.516\ldots$ and so $u = 1 - \frac{1}{m} = 0.340\ldots$

Polya: a drunk man will return home but a drunk bird is lost.