Recall: A process (X_n) is recurrent if it returns to its starting point w.p. 1

Transient if Prob(return) < 1.

Seen that for a RW, $u = P(\text{return to } \vec{0})$ related to $m = EN$ where $N = \#\text{returns} = \#\{n : \cancel{X}S_n = \vec{0}\}$

by $m = \frac{1}{1-u}$ $\quad u = 1 - \frac{1}{m}$

$$m = \sum_{n=0}^{\infty} P(S_n = \vec{0})$$

$$\left.\begin{array}{ll} \underline{d=1}: \; P(S_{2n} = 0) \sim \frac{1}{\sqrt{n\pi}} & \text{so } \sum P(S_n = 0) = \infty \\ \underline{d=2}: \; P(S_{2n} = \vec{0}) \sim \frac{1}{n\pi} & \text{so } \sum P(S_n = \vec{0}) = \infty \end{array}\right\} \text{Recurrent}$$

$N = \sum_{n=0}^{\infty} I_n$ where $I_n = \begin{cases} 1 & S_n = \vec{0} \\ 0 & \neq \vec{0} \end{cases}$

$E N = E\left(\sum I_n\right) = \sum (E I_n) = \sum P(S_n = \vec{0})$

In $d>2$: $P(S_{2n} = \vec{0}) \sim c n^{-d/2}$ so $\sum P(S_{2n} = \vec{0}) < \infty$

For vector R.V. $\vec{X}$ the char. func. is $\varphi(\vec{t}) = E e^{i\langle \vec{t}, \vec{X} \rangle}$

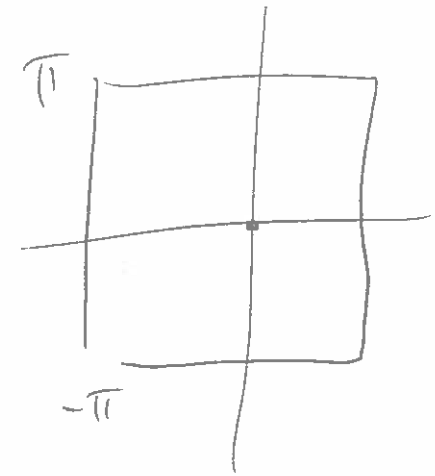
$\langle \vec{t}, \vec{X} \rangle = \sum X_i t_i$

If X has integer coord. then

$$P(\vec{X} = \vec{0}) = \left(\frac{1}{2\pi}\right)^d \iint_B \varphi(\vec{t})\, d\vec{t} \qquad B = [-\pi, \pi]^d$$

char. func of S_n is $[\varphi_X(\vec{t})]^n$

$$m = \sum_n P(S_n = \vec{0}) = \sum_n \left(\frac{1}{2\pi}\right)^d \left[\iint_B \varphi^n \, d\vec{t}\right]$$



$$= \left(\frac{1}{2\pi}\right)^d \iint \frac{1}{1-\varphi} \, d\vec{t}$$

--- calculate --- estimate ---

$m \leqslant \infty$ if $d > 2$

$m = \infty$ if $d \leq 2$

Markov Chains

A M.C. is a stochastic process; sequence (X_n) of R.V.s, which satisfies the Markov property:

Transition probab: $P(X_{n+1}=j \mid X_n=i)$ denoted p_{ij}

Markov property:

$$P(X_{n+1}=j \mid X_n=i) = P(X_{n+1}=j \mid X_n=i, X_0=i_0, X_1=i_1, \ldots, X_{n-1}=i_{n-1})$$

for every $n, i, j, i_0, \ldots, i_{n-1}$

X_n take values in a set S called state space e.g. Weather model

$S = \{\underbrace{\text{rain}}_{0}, \underbrace{\text{sun}}_{1}\}$

$P(X_{n+1} = \text{rain} \mid X_n = \text{rain}) = P(X_{n+1} = 0 \mid X_n = 0) = 0.4$

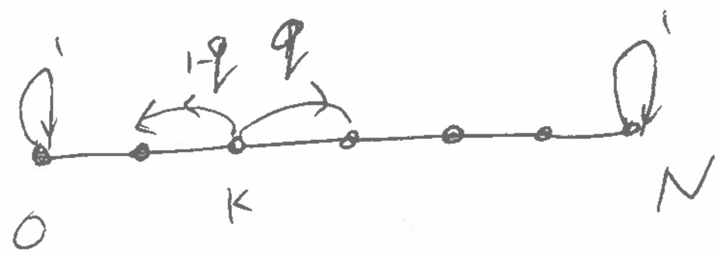
$P(X_{n+1} = 1 \mid X_n = 0) = 0.6 = 1 - 0.4$

$P(X_{n+1} = 0 \mid X_n = 1) = 0.2$ so $P(X_{n+1} = 1 \mid X_n = 1) = 0.8$

---

e.g. Gambler's ruin:

$S = \{0 \dots N\}$



$\left.\begin{array}{l} P_{k,k+1} = p \\ P_{k,k-1} = 1-p \end{array}\right\}$ for $0 < k < N$ $\quad P_{00} = P_{NN} = 1$

Note: We assume for this course that

$$P_{ij} = P(X_{n+1}=j \mid X_n=i)$$ does not depend on n.

These can be entered in a matrix

$$P = (P_{ij})_{i,j} \qquad \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & & & \end{pmatrix}$$

Order of the states
for rows + cols can be any order,
but must be same for rows and cols.

e.g. Gambler's ruin:

$$P = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & & & \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ \\ \\ \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1-q & 0 & q & 0 & 0 & 0 \\ & 1-q & 0 & q & & \\ & & 1-q & 0 & q & \\ & & & 1-q & 0 & q \\ 0 & & & & 1-q & 0 \;\; q \end{pmatrix} \end{array} \longleftarrow \quad P_{00} = 1 \qquad P_{0i} = 0 \quad i \neq 0$$

[Transition matrix for the Markov chain]