2023-03-15

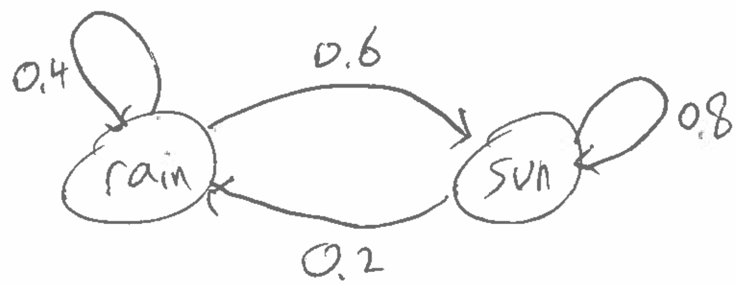
Recall $P_{ij} = P(X_{n+1}=j \mid X_n=i)$ trans. probab.

$= P(X_{n+1}=j \mid X_n=i, X_0=\cdots X_{n-1}=\cdot)$ [markov property

idea: conditioned on present state (X_n) the future state $(X_{n+1}, \ldots)$ are indep. of past states.

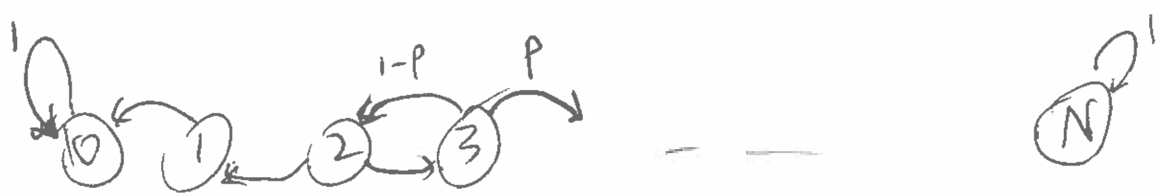
special case: (X_n) are indep.

e.g.



transition diagram

e.g.



Transition matrix:

$$P = \begin{pmatrix} p_{ij} \end{pmatrix}_{i,j \in S}$$

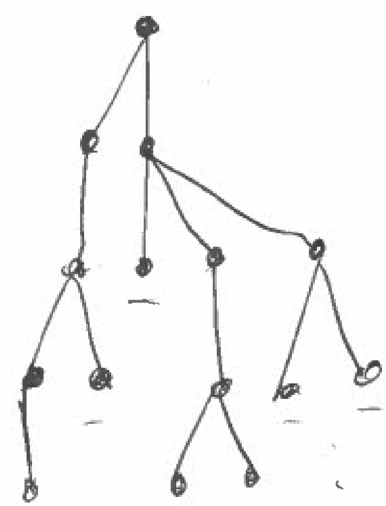
e.g. <u>Branching process</u>:

gen 0

gen 1

gen 2

gen 3



$X_0 = 1$

$X_1 = 2$

$X_2 = 4$

$X_3 = 5$

$X_4 = 3$

Each indiv. has indep. number of children with a given distrib.

note: If $X_n = 0$ then $X_{n+1} = 0$.

$P_{00} = 1$

0 is called an absorbing state:

state i with $P_{ii} = 1$

If a branching process reachs 0 we say it dies out.

n-step transition probab.:

what is: $P(X_{\ell+n}=j \mid X_\ell=i)$? $n=1$: P_{ij}

<u>note</u>: for $n=0$ this is $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

For $n=1$ this is P_{ij}

<u>notation</u>: $P^n_{ij} = P(X_{\ell+n}=j \mid X_\ell=i)$

<u>Claim</u>: For any $m,n \geq 0$ $\quad P^{n+m}_{ij} = \sum_k P^n_{ik} P^m_{kj} \qquad i,j,k \in S$

<u>proof</u>: $P(X_{\ell+n+m}=j \mid X_\ell=i) = \sum_k P(X_{\ell+n+m}=j, X_{\ell+n}=k \mid X_\ell=i)$

$$= \sum_k \underbrace{P(X_{\ell+n}=k \mid X_\ell=i)}_{P^n} \cdot \underbrace{P(X_{\ell+n+m}=j \mid X_{\ell+n}=k, X_\ell=i)}_{P^m}$$

Markov property: given $X_{\ell+n}$, the later states $(X_{\ell+n+m})$ are indep. of earlier states (X_ℓ)

So 2nd term in the product is $= P(X_{\ell+n+m} = j \mid X_{\ell+n} = k)$

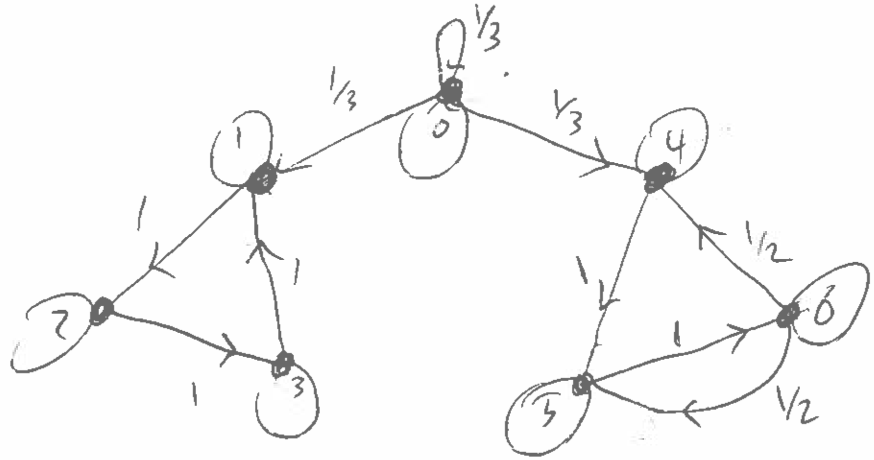
$$= P^m_{kj}$$

So $P^{n+m}_{ij} = \sum_k P^n_{ik} P^m_{kj}$ $\square$

The claim says that n-state trans. probabilities are $P^{n+m} = P^n \cdot P^m$ as a product of matrices.

$P^n =$ nth power of P as a matrix.

e.g.



Assume $X_0 = 0$.

communicating classes here are $\{0\}$ $\{1,2,3\}$ $\{4,5,6\}$

def: j is accessible from i, denote $i \to j$ if there is some $n \geq 0$ s.t. $P^n_{ij} \neq 0$

i and j communicate if $i \to j$ and $j \to i$

Claim: i communicates with j $(i \leftrightarrow j)$ is an equivalence relation.

Proof: If $i \longrightarrow j$ and $j \longrightarrow k$ then some n has $P^n_{ij} > 0$ and some m has $P^m_{jk} > 0$

$$P^{n+m}_{ik} = \sum_{\ell} P^n_{i\ell} P^m_{\ell k} \geq P^n_{ij} P^m_{jk} > 0$$

so $i \leftrightarrow j$ and $j \leftrightarrow k$ implies $i \leftrightarrow k$

$\square$

e.g. Gambler's ruin, comm. classes are

$\{1, \ldots N-1\}$ $\{0\}$ $\{N\}$

Absorbing state (with $P_{ii} = 1$) is always its own comm. class.