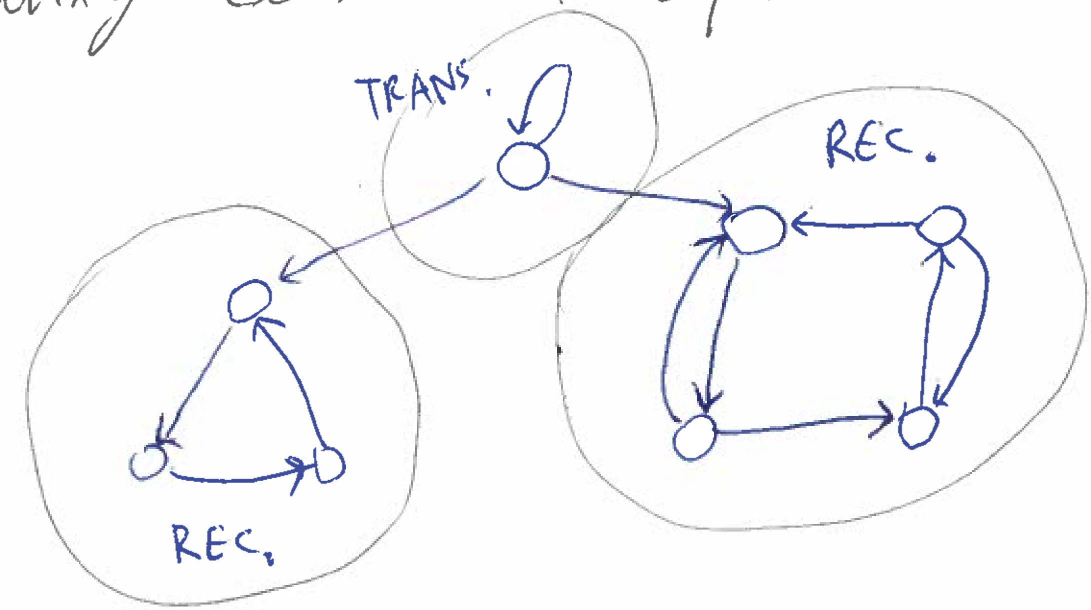
2023-03-1

<u>Recall</u>: i, j communicate if $\exists n, m$ s.t. $P^n_{ij} > 0$ $P^m_{ji} > 0$

denoted $i \leftrightarrow j$

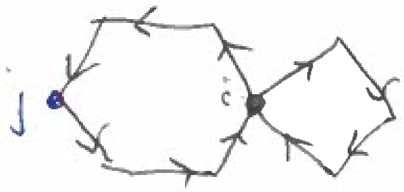
Communicating classes: Equivalence classes for $\leftrightarrow$.



<u>Periodicity</u>: A state i has period d if

$d = GCD\{n : P^n_{ii} > 0\}$

i periodic if $d > 1$ aperiodic if $d = 1$

e.g. 

$P^n_{ii} > 0$ for $n = \{4, 6, 8, 10, 12, \ldots\}$

GCD = 2.

$P^n_{jj} > 0$ for $n = \{6, 10, 12, 14, \ldots\}$ GCD = 2

<u>Claim</u>: If $i \leftrightarrow j$ then i and j have the same period.

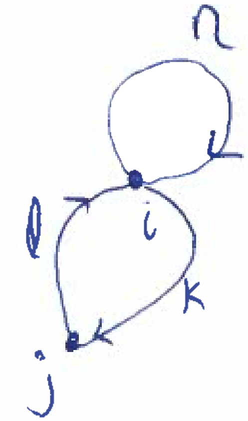
$\Longrightarrow$ Every comm. class of a M.C. has a period

<u>Def</u>: If every i, j communicate, then the M.C. is called irreducible.

idea of proof:

Assume $P_{ij}^{k} > 0$ and $P_{ji}^{l} > 0$.

If $P_{ii}^{n} > 0$ then $P_{jj}^{n+k+l} > 0$.



Claim: $P_{ii}^{n} > 0$ for every large enough multiple of the period. ($\forall i\, \exists K\, \forall n > K \ldots$)

so $\{n : P_{ii}^{n} > 0\}$ and add $k+l$ to each, is contained in $\{n : P_{jj}^{n} > 0\}$ so periods are equal $\square$

Recurrence + Transience

def: Let $f_i = P(\exists n>0 \text{ s.t. } X_n = i \mid X_0 = i)$

A state i is recurrent if $f_i = 1$,

transient if $f_i < 1$.

Proposition: If $i \leftrightarrow j$ then both are recur. or both are transient.

[recurrence and transience are class properties]

Let N_i be the total number of visits to i if $X_0 = i$

If $f_i = 1$ then $N_i = \infty$.

If $f_i < 1$ then $P(N_i = k) = \underbrace{f_i \cdot f_i \cdot f_i}_{k-1} (1-f_i) = f_i^{k-1}(1-f_i)$

so $N_i \sim \text{Geom}(1-f_i)$ so $EN_i = \frac{1}{1-f_i}$

$N_i = \sum_n 1_{\{X_n = i\}}$ so $EN_i = \sum_{n=0}^{\infty} P(X_n = i \mid X_0 = i) = \sum_{n=0}^{\infty} p_{ii}^n$

i is recurrent $\iff \sum_n p_{ii}^n = \infty$.

If $p_{ij}^k > 0$ and $p_{ji}^l > 0$, then $p_{ii}^{k+n+l} \geq p_{ij}^k \, p_{jj}^n \cdot p_{ji}^l$

$$\sum_n p_{ii}^{k+n+l} \geq p_{ij}^k \, p_{ji}^l \sum_n p_{jj}^n$$

If j is recurrent then $\sum_n p^n_{jj} = \infty$ so $\sum_n p^n_{ii} = \infty$

$$\left[\sum_n p^n_{ii} = \sum_{n<k+l} p^n_{ii} + \sum_n p^{k+l+n}_{ii}\right]$$

so i is recurr. $\square$

<u>note</u>: If a M.C. is irreducible, we say the M.C. is recurrent / transient.

<u>note</u>: If S is finite then at least one state is recurrent.

state space