2023-03-2

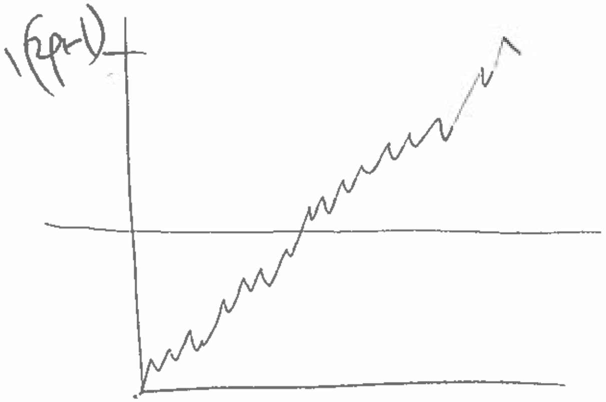
Note on recurrence

T = return time to 0 $= \min\{n>0 : X_n = 0\}$

for a RW on $\mathbb{Z}$.

$$P(T=\infty) = \begin{cases} 0 & \text{no bias} \\ >0 & \text{bias} \quad p \neq 1/2 \end{cases}$$

$p > 1/2$



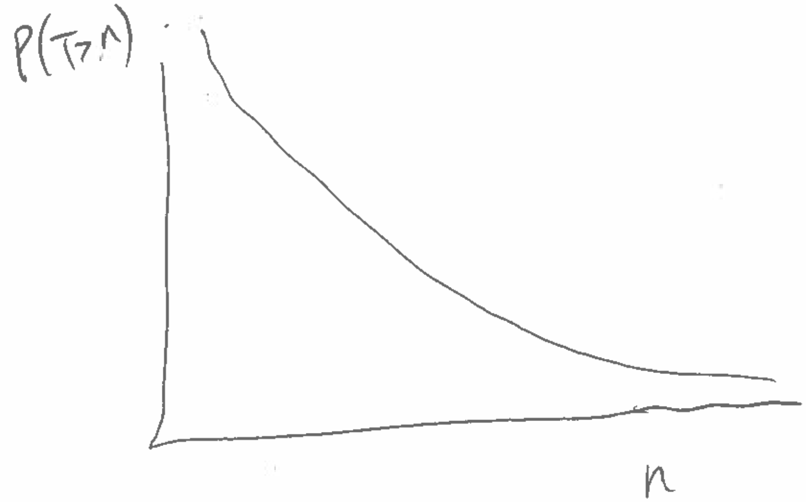
avg time at each level is $\frac{1}{2p-1}$

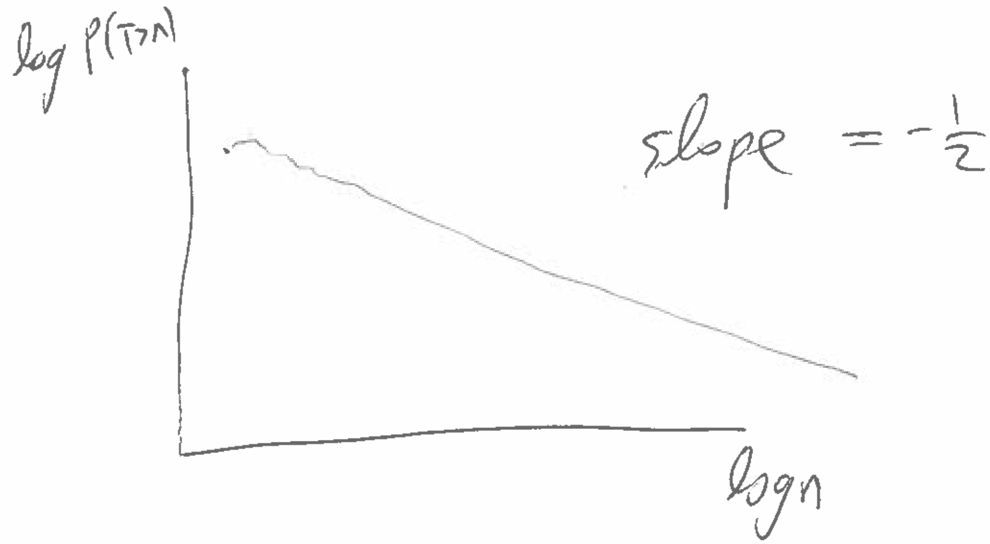
$\left[\frac{1}{2p-1} = m = \frac{1}{1-u} \quad \text{so} \quad u = 2-2p \right]$

LLN: $S_n \sim n(2p-1)$

If $p = \frac{1}{2}$, $T < \infty$ always, but can be quite large

$P(T > n) \approx \frac{c}{\sqrt{n}}$ $\qquad$ $ET = \infty$



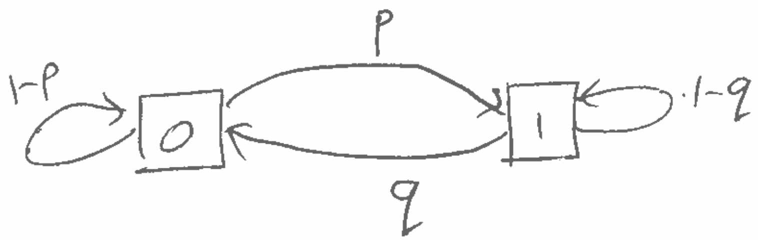


$\log P(T > n) \approx c' - \frac{1}{2} \log n$

<u>Note</u>: In $\mathbb{Z}^2$ $\quad P(T > n) \approx \frac{c}{\log n}$

Stationary distributions

e.g. machine is working (1) or broken (0).



$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

Assume X_0 has distrib. $v^{(0)}$: $P(X_0 = i) = v^{(0)}_i$

What is the distrib. of X_n denoted $v^{(n)}$?

n-step trans probab are $P^n_{ij} = P(X_n = j \mid X_0 = i)$

so $P(X_n = j) = \sum_i P(X_0 = i) \cdot P(X_n = j \mid X_0 = i)$

$$= \sum_i v^{(0)}_i P^n_{ij} = (v^{(0)} P^n)_j$$

<u>Summary</u>: $v^{(n)} = v^{(0)} P^n$ $\qquad$ v is $1 \times |S|$ vector

P is $|S| \times |S|$

If X_0 known to be i then $v^{(0)} = (0, \ldots, 0, 1, 0, \ldots, 0)$

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e.g. 2-state M.C., with $X_0 = 0$:

$$v^{(n)} = (1, 0) \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}^n$$

e.values: λ s.t. $P - \lambda I$ is singular:

$$0 = \det(P - \lambda I) = \begin{vmatrix} 1-p-\lambda & p \\ q & 1-q-\lambda \end{vmatrix}$$

$\lambda = 1$ or $\lambda = 1 - p - q$

e. vectors: x s.t. $xP = \lambda x$

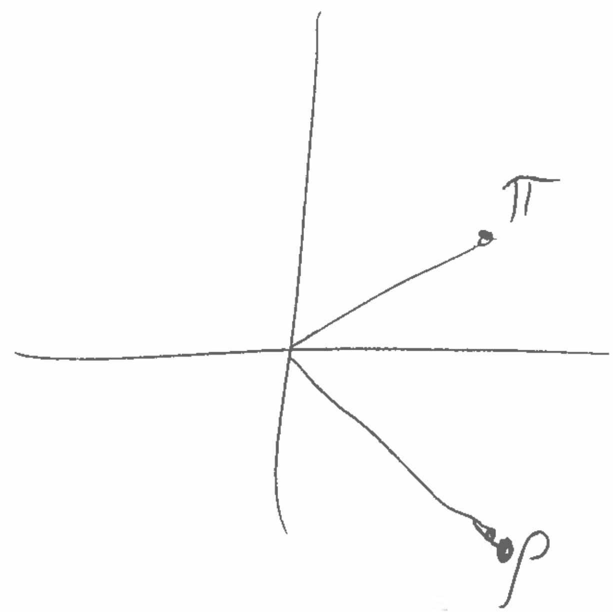
$$(1\ \ -1)\begin{pmatrix}1-p & p\\ q & 1-q\end{pmatrix} = (1-p-q\ ,\ \ p+q-1) = (1-p-q)\,(1\ \ -1)$$

$$(q\ \ p)\begin{pmatrix}1-p & p\\ q & 1-q\end{pmatrix} = (q\ \ p)$$ so e vectors are

$\pi = (q\ \ p)$ $\quad \rho = (1\ \ -1)$

$V = a\pi + b\rho$ for some a, b

$$V^{(0)}P^n = a\pi P^n + b\rho P^n$$
$$= a\pi + b\rho(1-p-q)^n$$



$\pi P^n = \pi$

$\rho P = \rho(1-p-q)$

$|1-p-q| < 1$ unless $p=q=0$ or $p=q=1$

so $(1-p-q)^n \xrightarrow[n\to\infty]{} 0$.

So $v^{(n)} \xrightarrow[n\to\infty]{} a\pi$

Sum of entries is 1 so $a = \frac{1}{p+q}$.

For any $v^{(0)}$ we have $v^{(n)} \xrightarrow[n\to\infty]{} \underbrace{\left(\frac{q}{p+q} \quad \frac{p}{p+q}\right)}$

This is the limit distrib. It gives the fraction of time at each state in the long run.

Def: A recurrent state i is positive recurrent if $ET < \infty$

[T = return time] null recurrent if $ET = \infty$

motivation: M.C. spends a positive fraction of time at a pos. recur. state

e.g. RW on $\mathbb{Z}$ or $\mathbb{Z}^2$ has null recurrent states

e.g. If $|S| < \infty$ then only pos. recur. is possible.

Def: A M.C. is ergodic if it is ~~irre~~ aperiodic and positive recurrent.

Thm: (*) For an irreducible ergodic M.C. there is a vector π s.t. $P(X_n=j) \xrightarrow[n\to\infty]{} \pi_j$ for any $v^{(0)}$

$[\, P^n_{ij} \xrightarrow[n\to\infty]{} \pi_j \quad \forall i,j \,]$

(*) If $N_j(T)$ = # visits to state j by time T

then $\frac{N_j(T)}{T} \xrightarrow[T\to\infty]{} \pi_j$

(*) π is the unique solution to $\pi P = \pi$ and $\sum \pi_i = 1$

i.e. $\pi_j = \sum_i \pi_i P_{ij}$

π is called the ~~statqionary~~ stationary distrib. for the M.C.

<u>note</u>: $P\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ since P has row sums $= 1$

so P also has e.vector with $\pi P = \pi$