2023-03-22

Stationary Distributions (Steady state, Equilibrium)

<u>def</u>: π is a stat. distrib for a M.C. with transition matrix P if $\underbrace{\pi P = \pi}_{\forall j \ \pi_j = \sum \pi_i P_{ij}}$ and $\sum \pi_i = 1$ (and $\pi_i \geq 0$).

<u>notes</u>: ⊛ A sol. always exists for finite S; since P is stochastic (row sums are 1) then 1 is an e. value.

⊛ If P irreducible then π is unique, and has $\pi_i > 0$

⊛ If P not irreducible, can have multiple S.D.s

⊛ If X_n has dist. π : $P(X_n = i) = \pi_i \quad \forall i$

then X_{n+1} also has dist. π.

<u>Proof</u>: $P(X_{n+1} = j) = \sum_i P(X_n = i)\, P(X_{n+1} = j \mid X_n = i)$

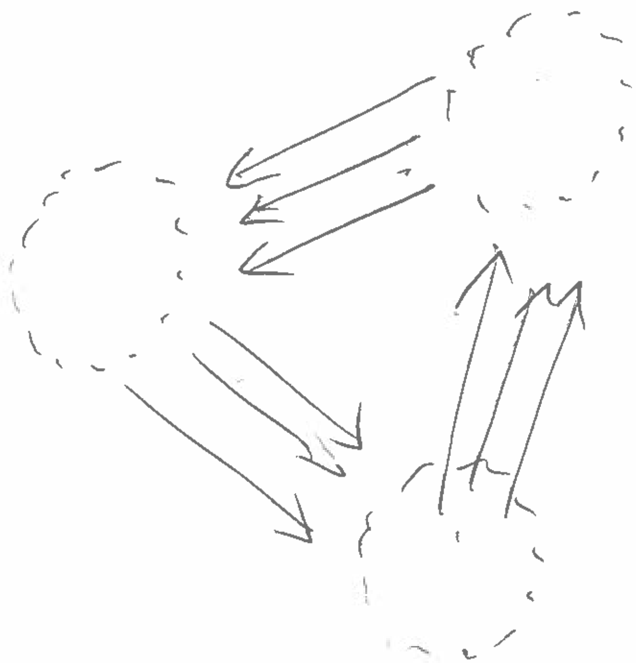
$$= \sum_i \pi_i P_{ij} = \pi_j$$

⊛ If P is aperiodic, and irreducible, [P still finite]

then for any i, $P_{ij}^{n} \xrightarrow[n \to \infty]{} \pi_j$

[If $X_0 = i$ then the distrib. of $X_n \xrightarrow[n \to \infty]{} \pi$]

⊛ If P periodic then p_{ij}^{n} does not converge but does converge if $n \equiv \text{const} \mod (d)$ where $d =$ period.



Period 3

If P has ∞ state space, might not have a stat. distrib.

e.g. RW on $\mathbb{Z}$ or $\mathbb{Z}^d$ $\quad P_{ij}^n \xrightarrow[n\to\infty]{} 0 \quad \forall i,j$

If a M.C. is transient then $\sum P_{ii}^n < \infty$ so

$P_{ii}^n \xrightarrow[n\to\infty]{} 0$

Can also get $P_{ii}^n \longrightarrow 0$ in a recurrent M.C.

<u>Recall</u>: M.C. is <u>positive recurrent</u> if $E T_i < \infty$

where T = return time to i.

null recurrent : $E T_i = \infty$ but $T < \infty$

A pos. recur. M.C. behaves like a finite state space M.C.:

A stat. distrib. exists, $P_{ij}^{n} \xrightarrow[n\to\infty]{} \pi_j$ if aperiodic

---

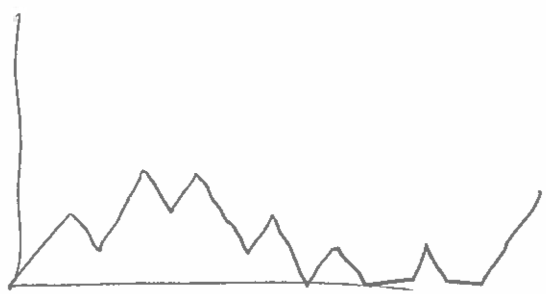
e.g. RW on $\mathbb{Z}$ with drift towards 0.

e.g. Length of a Queue:

$$\left.\begin{aligned} P_{n,n+1} &= q \\ P_{n,n-1} &= 1-q \end{aligned}\right\} n>0$$

$$P_{0,1} = q$$

$$P_{0,0} = 1-q$$



state space $= \mathbb{N} = \{0,1,2,\dots\}$

If $q > \frac{1}{2}$ this is transient. $(X_n \sim n(2q-1))$

(roughly this is the LLN + extra argument)

If $q \leq \frac{1}{2}$ this is recurrent.

<u>Finding π here:</u> $\pi_j = \sum_i \pi_i P_{ij} \quad \forall j$

$\pi_0 = \pi_0 P_{00} + \pi_1 P_{10} = \pi_0 (1-q) + \pi_1 (1-q)$ $\qquad q\pi_0 = \pi_1(1-q)$

$\pi_1 = \pi_0 P_{01} + \pi_2 P_{21} = \pi_0 q + \pi_2 (1-q)$ $\qquad (\pi_1 - \pi_0) q = (\pi_2 - \pi_1)(1-q)$

$\pi_2 = \pi_1 q + \pi_3 (1-q)$ $\qquad (\pi_n - \pi_{n-1}) q = (\pi_{n+1} - \pi_n)(1-q)$

$\vdots$

For any n $\quad \pi_{n+1} - \pi_n = \beta(\pi_n - \pi_{n-1})$ $\quad$ where $\beta = \frac{q}{1-q}$

$$= \beta^n (\pi_1 - \pi_0)$$

$$-\pi_n = \lim_{T\to\infty} (\pi_T - \pi_n) = \lim \sum_n^{T-1} \pi_{k+1} - \pi_k = \sum_n^{\infty} \pi_{k+1} - \pi_k$$

$$= \sum_{k=n}^{\infty} (\pi_1 - \pi_0) \cdot \beta^k = (\pi_1 - \pi_0)\beta^n / 1-\beta$$

π_n prop. to β^n and since $\sum \pi_n = 1$ get $\pi_n = \frac{\beta^n}{1-\beta}$

<u>Summary</u>: stat. distrib. is $\text{Geom}(1-\beta) - 1$

no stat. distrib. if $q \geq 1/2$.

$q = 1/2$ is null recurrent

Thm: If $X_0 = i$ then $E\,T_i = E$ return time $= \frac{1}{\pi_i}$

If an avg return every m steps $m = E\,T_i$ then

fraction of time at i is $\frac{1}{m}$