TIME REVERSAL

Given a MC $X_1, X_2, \dots,$ can look at the backwards chain by reversing time.

<u>Theorem</u>: Given a MC w/ stationary distrib. π, and $\boxed{P(X_0 = j) = \pi_j}$, let $Y_n = X_{\underline{N}-n}$ be the reversed chain. Then

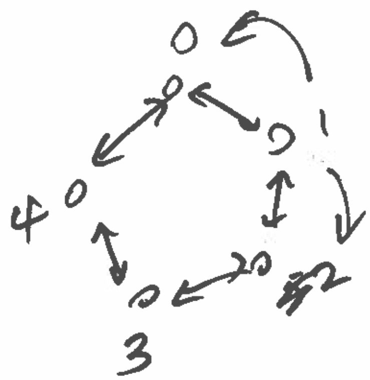
(1) Y is a M.C.

(2) Y has transition probs $Q_{ij} = P_{ji} \frac{\pi_j}{\pi_i}$

(3) Y has stationary dist. π.

"random walk"

ex RW on n-cycle $(s \in (0,1))$



$$X_{n+1} = X_n + \varepsilon_n \quad (\text{mod } n),$$

ε_n are iid w.p. $P(\varepsilon_n = 1) = s$
$P(\varepsilon_n = -1) = 1-s$

Q: what is the stat distrib?

$$P_{ij} = \begin{cases} 0, & |i-j| \neq 1 \quad (\text{mod } n) \\ s, & j = i+1, \quad (\text{mod } n) \\ 1-s, & j = i-1, \quad (\text{mod } n) \end{cases}$$

$\pi = \frac{1}{n}(1, 1, \ldots, 1)$ is stat.

Check: $(\pi P)_j = \frac{1}{n} P_{j-1,j} + \frac{1}{n} P_{j+1,j} = \frac{1}{n}(s + 1 - s)$
$= \frac{1}{n}$ ✓

Guess: time reversal is same chain but with s swapped w/ $1-s$.

Check directly by computing Q_{ij}.

- **Def** A M.C. time-reversible if

$P_{ij} = Q_{ij}$ for all i,j. (Note: this includes the fact that Q is well-defined.)

Equiv to $P_{ij} = Q_{ij} = P_{ji} \frac{\pi_j}{\pi_i}$

$$\Longleftrightarrow \boxed{\pi_i P_{ij} \overset{(*)}{=} \pi_j P_{ji}. \text{ for all } i,j}$$

"detailed balance eqs."

Theorem: If there is a solution $\vec{x}$ to the (*) eqs, i.e. $x_i p_{ij} = x_j p_{ji}$ $\forall i,j$ and $\sum_j x_j = 1$, then $\vec{x} = \pi$ is the stat. dist. and the chain is ~~...~~ time-reversible.

Proof: If (*) eqs hold for $\vec{x}$, then

check that $\vec{x} P = \vec{x}$.

$$(\vec{x} P)_j = \sum_i x_i p_{ij} \overset{(*)}{=} \sum_i x_j p_{ji}$$

$$= x_j \cancelto{1}{\sum_i p_{ji}}$$

$$= x_j$$

So $x = \pi$ is stat., and so the chain is time reversible. (* is satisfied with π).

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e.g. RW on $\{0, 1, 2, 3, \dots\}$, $s < \frac{1}{2}$.

$$p_{i,i+1} = s, \quad p_{i,i-1} = 1-s \quad i \geq 1$$

$$p_{0,1} = s, \quad p_{0,0} = 1-s$$

idea: use (*) to find π.

Guess: $\pi_i = \underline{r^i (1-r)}$, some $r \in (0,1)$.

$$\Rightarrow \pi_i p_{i,i+1} \overset{?}{=} \pi_{i+1} p_{i+1,i}$$

$$r^i (1-r) s \overset{?}{=} r^{i+1} (1-r)(1-s)$$

$$\Rightarrow r = \frac{s}{1-s}$$ for this to possibly be a solution.

Check $i=0$: $\quad \pi_0 p_{01} \stackrel{?}{=} \pi_1 p_{10}$

$\pi_i = r^i(1-r)$ $\qquad$ $(1-r)s \stackrel{?}{=} r\cancel{(1-r)}(1-s)$

$r = \frac{s}{1-s}$

What about $i=j=0$?
Always holds.

So $\pi_i = r^i(1-r)$ is stat., and the chain is time reversible.

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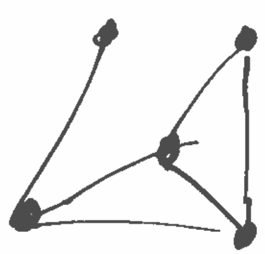
ex. SRW on a graph (simple random walk)

Graph G, vertices V and edges $E = \{(v,w):$ some pairs $v,w \in V$



& $P_{v,w} = \begin{cases} \frac{1}{\deg v}, & (v,w) \in E \\ 0, & \text{oth.} \end{cases}$

Seen before stat dist. $\pi_v = \dfrac{\deg v}{\sum_{v' \in V} \deg v'}$

Check (*): $\quad \pi_v p_{vw} \stackrel{?}{=} \pi_w p_{wv}$ $\quad$ (if $(v,w) \in E$)

Is time-reversible $\quad \dfrac{\cancel{\deg v}}{\sum \deg} \cdot \dfrac{1}{\cancel{\deg v}} = \dfrac{\cancel{\deg w}}{\sum \deg} \dfrac{1}{\cancel{\deg w}}$