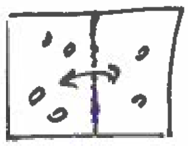
(Wednesday, March 29)

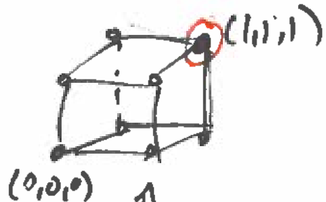
<u>example</u> (Ehrenfest chain)

(SRW on the hypercube) $\leftarrow$ state space $\{0,1\}^M$

$= \{(x_1, x_2, \dots, x_M) : x_i \in \{0,1\}\}$



M particles




$\uparrow$ M-dimensional

Transitions: pick a random index $i \in \{1, 2, \dots, M\}$, and flip the bit.

Study $X_n$ = # particles on left side of the barrier

$= \sum_{i=1}^{M} x_i$ ($x_i = 1$ if particle $i$ on the left)

$X_n$ is a MC on $\{0,1,2,\dots M\}$.

Find $\pi$; is $X$ reversible?

Solve for $\pi$ using detailed balance.

Notation: $p_{i,i+1} = \frac{M-i}{M} = \alpha_i$ $\quad (p_{i,i-1} = 1 - \alpha_i)$

(detailed balance w/ $i=0$, $j=1$)

$\pi_0 \alpha_0 = \pi_1 (1-\alpha_1) \Rightarrow \pi_1 = \pi_0 \cdot \frac{\alpha_0}{1-\alpha_1}$

$\pi_1 \alpha_1 = \pi_2 (1-\alpha_2) \Rightarrow \pi_2 = \pi_1 \cdot \frac{\alpha_1}{1-\alpha_2} = \frac{\alpha_0 \alpha_1}{(1-\alpha_1)(1-\alpha_2)}$

$\vdots$

$$\pi_i = \frac{\alpha_{i-1}\alpha_{i-2}\cdots\alpha_0}{(1-\alpha_i)(1-\alpha_{i-1})\cdots(1-\alpha_1)}\pi_0 .$$

Plug $\alpha_i = \frac{M-i}{M}$, $1-\alpha_i = \frac{i}{M}$

$$\Rightarrow \quad = \frac{\frac{1}{M^i}(M-i+1)(M-i+2)\cdots M}{\frac{1}{M^i}\, i\,(i-1)(i-2)\cdots 2\cdot 1}\pi_0$$

$$= \binom{M}{i}\pi_0 . \left( = \frac{M!}{i!\,(M-i)!}\pi_0 \right)$$

Need $\sum_j \pi_j = 1 \Rightarrow \pi_0 = 2^{-M}$,

$$\pi_i = 2^{-M}\binom{M}{i}.$$

So $\pi \sim \mathrm{Bin}(M, \frac{1}{2})$.

$X$ is reversible, and $\pi$ is stat.
(By thm from last class.)

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**Q:** in general, what irreducible MCs on a finite state space are reversible?

**A:** Iff detailed balance eqs have a solution.

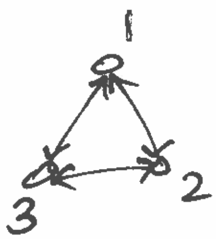
"Typically" no solutions:

$$\pi_i p_{ij} = \pi_j p_{ji} \quad - \quad n \text{ states}, \; n^2 \text{ equations}$$

Solving for $\pi = (\pi_1, \ldots, \pi_n)$. $n$ unknowns

e.g. not reversible if $p_{ij} > 0, p_{ji} = 0$ for some pair of states $i \neq j$.

e.g. $n = 3$. Arb. MC on $\{1, 2, 3\}$.




$$\pi_1 p_{12} = \pi_2 p_{21} \longrightarrow \pi_2 = \frac{p_{12}}{p_{21}} \pi_1$$

$$\pi_2 p_{23} = \pi_3 p_{32} \longrightarrow \pi_3 = \frac{p_{23}}{p_{32}} \pi_2 = \frac{p_{12} p_{23}}{p_{21} p_{32}} \pi_1$$

$$\pi_3 p_{31} = \pi_1 p_{13} \longrightarrow \pi_1 = \frac{p_{31}}{p_{13}} \pi_3 = \frac{p_{12} p_{23} p_{31}}{p_{13} p_{32} p_{21}} \pi_1$$

$$\Longrightarrow \quad p_{12} p_{23} p_{31} = p_{13} p_{32} p_{21}$$

$P_1$(one clockwise cycle) = $P_1$(one cc-wise cycle)

**Theorem**: if for *every* cycle $(v_0, v_1, v_2, \ldots, v_r =$

$$P(X_0 = v_0, X_1 = v_1, \ldots, X_r = v_r = v_0)$$

$$\overset{(*)}{=}$$

$$P(X_0 = v_0, X_1 = v_{r-1}, X_2 = v_{r-2}, \ldots, X_{r-1} = v_1, X_r = v_0),$$

then $X$ is time reversible.

(Converse holds: if $X$ is reversible, then $(*)$ holds for every cycle.)

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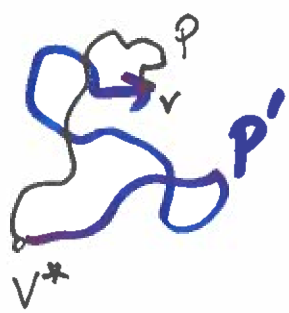
**Pf**: (sketch) Construct $\pi$: pick a root vertex $v^*$. Define $\pi_v$ by:




Pick path $P$, define

$$\pi_v = \frac{\text{probstuff}}{\text{probstuff}} \cdot \pi_0 .$$

↑ "product along path $P$".

Use the cycle condition to show that this definition doesn't depend on which path $P$ you chose.

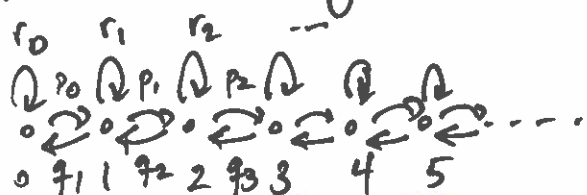
(Friday, March 31)

From last time: if the cycle condition(*) is satisfied, then the MC is reversible.

$$(*)\quad P(X_1 = v_1, \ldots, X_r = v_r = v_1)$$
$$= P(X_1 = v_1, X_2 = v_{r-1}, \ldots, X_{r-1} = v_2, X_r = v_1)$$

for **any** cycle $(v_1, v_2, \ldots, v_r)$, with $v_r = v_1$.

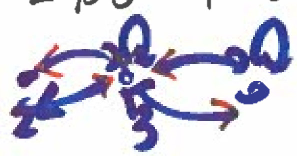
Application: any birth/death chain that has a stat. prob. measure is reversible.



$r_i = p_{i,i}$

$p_i = p_{i,i+1}$

$q_i = p_{i,i-1}$



cycle $(2, 3, 4, 4, 3, 3, 2)$.

- In this case, $\pi_i = \underbrace{\dfrac{p_0 p_1 p_2 \cdots p_{i-1}}{q_1 q_2 \cdots q_i}}_{A_i} \pi_0$ (directly from detailed balance eqs.) $A_0 = 1$.

Fact: if $\sum_{i \geq 0} A_i \begin{cases} < \infty, & \text{then chain is + recurrent} \\ = \infty, & \text{then " is either null rec. or transient.} \end{cases}$

ex $\quad p_i = p, \quad q_i = q, \quad r_i = 0$ $\qquad$ ($\pi$ = stat. measure)

Then $A_i = (p/q)^i, \quad \pi_i = (p/q)^i \pi_0. \qquad \pi P = \pi$

$$\sum A_i < \infty \text{ iff } p < q.$$

$\pi$ is geometric if $p < q$.

ex $\quad p_i = \frac{\lambda}{i+1+\lambda}, \quad q_i = \frac{i}{i+\lambda}, \quad r_i = \ldots \qquad (p_i + q_i + r_i = 1)$

$\lambda \in (0, \infty)$

Then $A_i = \frac{\lambda^i}{i!}$, so $\pi \sim \text{Poisson}(\lambda)$.

(M/M/$\infty$)
queue

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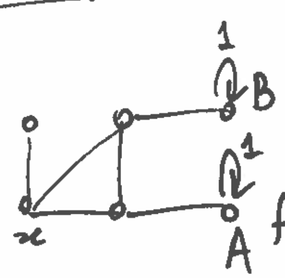
## Absorption probabilities / boundary value problems.




M.C. $X$ with absorbing states $A, B$

$f(x) = P(X_0 = x, \text{ and } \tau_A < \tau_B)$, where

"Tau"

$\tau_A = \min\{t \geq 0 : X_t \in A\}$ (same for B).
"hitting time"

($P_{xy}$) transition matrix.

$$\boxed{f(x) = \sum_y P_{xy} f(y), \text{ for all } x} \quad + \quad \begin{aligned} f(A) &= 1 \\ f(B) &= 0 \end{aligned}$$

"condition on first step"

Can solve this linear system for $(f(x))_{v \in V}$.

Similar Q: what is $\mathbb{E}[\tau_A]$? (expected # steps before hitting $A$).

In general, $A$ could be a set of states. ex SRW on $\{1, \dots, N\}$.

$$1 \overset{1/2}{\longleftrightarrow} 2 \overset{1/2}{\longleftrightarrow} 3 \overset{1/2}{\longleftrightarrow} 4 \overset{1/2}{\longleftrightarrow} 5 \quad \cdots \quad N$$

$A = \{1, N\}$.

Similar idea: let $g(x) = \mathbb{E}_x[\tau_A] = \mathbb{E}[\tau_A \mid X_0 = x]$.

Similar system of eqs relating the $g(x)$'s:

$$g(x) = 1 + \sum_y p_{xy}\, g(y). \quad \text{for all } x.$$

$+\ g(x) = 0$ for $x \in A$.

$$g(x) = 1 + \tfrac{1}{2} g(x+1) + \tfrac{1}{2} g(x-1).$$

$$g(x) = x(N-x).$$

(hitting prob for $A = 1$, $B=$

$f(x) = \frac{1}{2} f(x+1) + \frac{1}{2} f(x-$