First step Analysis

Idea: seporate a M.C. to first step + rest.

e.g. Let A be the event that a M.C. visits some state x.

(e.g. Gambler's ruin)

Let $f(u) = P(A \mid X_0 = u)$. Get eqn.s for $f(u)$ over all states.

$$\begin{cases} f(x) = 1 \\ f(u) = \sum_v P_{uv} \cdot f(v) \quad \text{for } u \neq x \end{cases}$$

e.g. $\mathbb{Z}^3$, $A = \{\text{hit } \vec{0}\}$.

$f(\vec{x}) = P(A \mid X_0 = \vec{x})$

f satisfies $f(\vec{0}) = 1$ $f(u) =$ avg of f at 6 neigh.

One sol. to these eqn.s is $f \equiv 1$

However, the RW is transient, so must have $f(x) \xrightarrow[x \to \infty]{} 0$

Can "solve" to get $f(\vec{x}) \approx \frac{c}{|x|}$

Hitting time: Let $\tau_A =$ first time that $X_n \in A$

for some set $A \subset S$.

e.g. Gambler's ruin: $A = \{0, N\}$.

Idea: Let $g(u) = E(T_A \mid X_0 = u)$ for any $u \in S$

If $u \in A$ then $g(u) = 0$

Otherwise, $g(u) = \underset{\substack{\uparrow \\ \text{first step}}}{1} + \sum_v \underbrace{P_{uv}\, g(v)}_{\text{rest if jump } u \to v}$

In gambler's ruin: $A = \{0, N\}$

$$\left.\begin{array}{l} g(0) = g(N) = 0 \\ (*)\ g(k) = 1 + \frac{1}{2}\big(g(k-1) + g(k+1)\big) \end{array}\right\} \Longrightarrow \begin{array}{l} g(k) = k(N-k) \\ \text{(unique sol.)} \end{array}$$

To find this: rewrite $(*)$ as

$$\big(g(k+1) - g(k)\big) = \big(g(k) - g(k-1)\big) - 2$$

so Δg is linear
$g(k)$ is quadratic.

$$
\begin{tikzcd}
A & B \arrow[l, "1/2"'] \arrow[r, "1/2"] & C \arrow[l, bend left, "2/3"] \arrow[loop right, "1/3"]
\end{tikzcd}
$$

If $X_0 = B$, what is the avg. time to hit A.

$$
\left.\begin{aligned}
g(A) &= 0 \\
g(B) &= 1 + \tfrac{1}{2} g(A) + \tfrac{1}{2} g(C) \\
g(C) &= 1 + \tfrac{1}{3} g(C) + \tfrac{2}{3} g(B)
\end{aligned}\right\} \text{ solve to find } g(B) \text{ and } g(C).
$$

Branching Processes

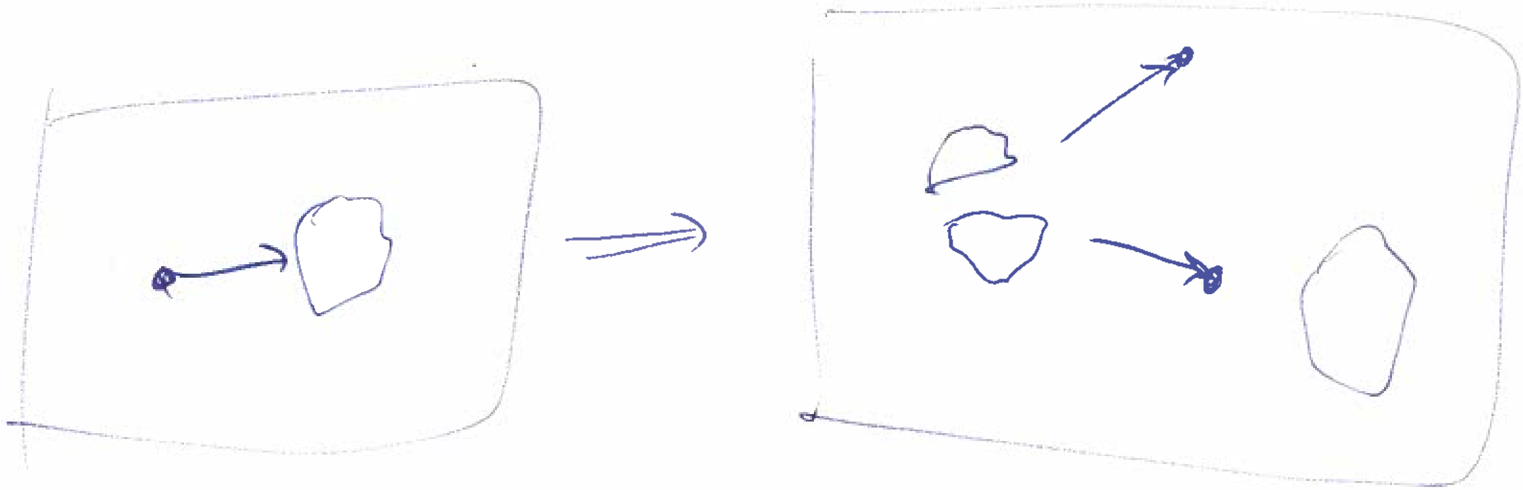
ξ is some R.V. taking values in $\mathbb{N}$.

Each indiv. has independently a copy of ξ children

\xi

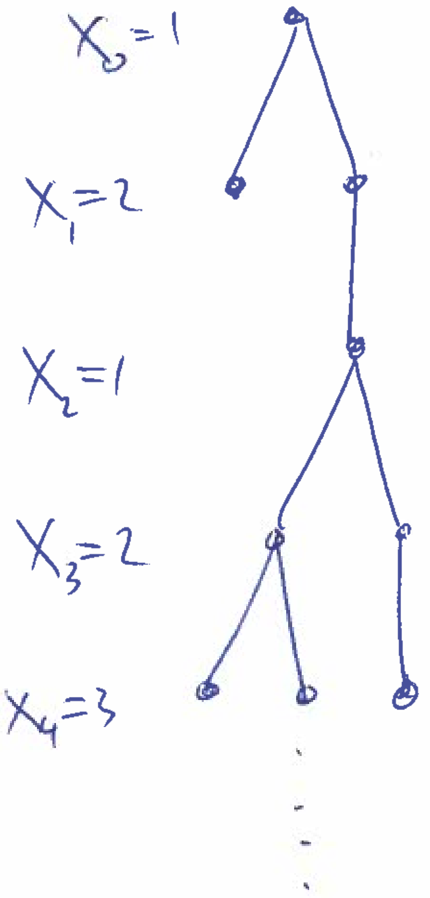
v has ξ_v children

e.g. ξ is $Bin(2, p)$ related to chain reactions.



<u>Qn</u>: If start with 1 indiv., how many are in gen. n?

(X_n) = size of gen. n.

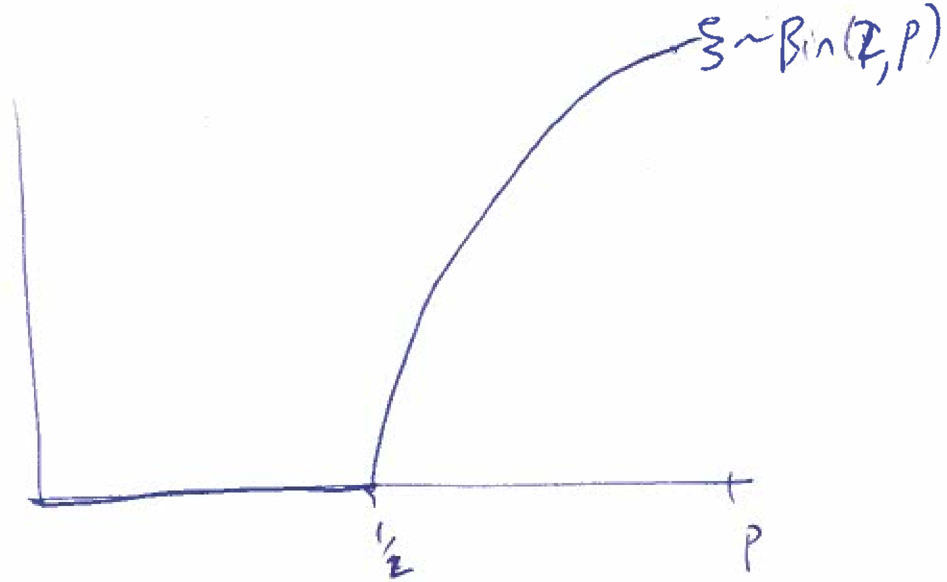


If $X_n=0$ then $X_{n+1}=0$ as well.

Extinction: $X_n=0$ for some n.

Survival: $X_n>0$ for all n.

P(survival)



given X_m, what is $E(X_{m+1} \mid X_m)$?

Ans: $E(X_{m+1} / X_m) = X_m \cdot E(\xi)$

So (by induction): $E(X_m) = (E\xi)^m$

<u>Thm</u>: If $E\xi \leq 1$ and $p(\xi = 1) < 1$ then $p(\text{survival}) = 0$

If $E\xi > 1$ then $p(\text{survival}) > 0$.

Probab. generating func.:

$g(s) = E\, s^X$ for R.V. X (used mostly if X takes values in $\mathbb{N}$)

$\varphi(x) = E\, e^{itX} = E(e^{it})^X = g(e^{it})$ relates to char. func.

If $X \in \mathbb{N}$ then $g(s) = p(0) + p(1)\cdot s + p(2)\cdot s^2 + \dots$

e.g. X is $\mathrm{Bin}(2,p)$ then $g(s) = (1-p)^2 + 2p(1-p)s + p^2 s^2$

$= (1-p+ps)^2$

$X \sim \mathrm{Bin}(n,p)$ then $g(s) = (1-p+ps)^n$

$X \sim \mathrm{Poi}(\lambda)$ $g(s) = e^{-\lambda(1-s)}$

Let $q_n = P(X_n = 0 \mid X_0 = 1)$ for a Branch. process

Let g be the prob. gen. func. for ξ

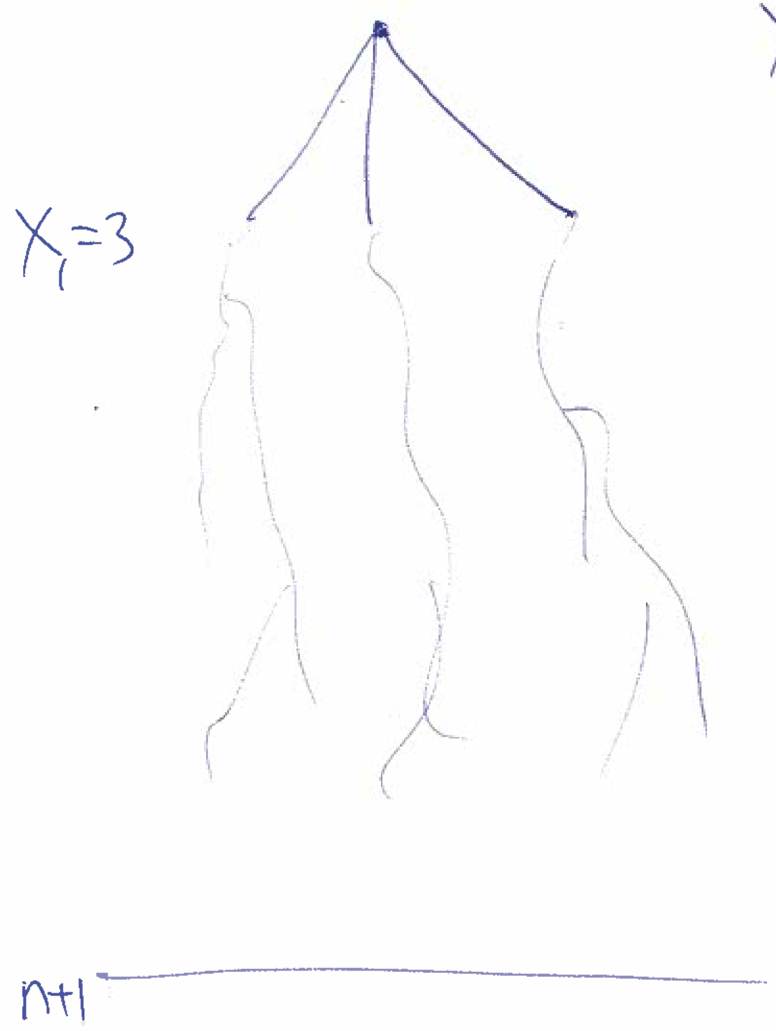
<u>Lemma</u>: $q_{n+1} = g(q_n)$ for all n

$q_0 = 0$

<u>note</u>: • $P(\text{extinction}) = \lim_{n\to\infty} q_n$

• q_n are increasing in n.

<u>Proof</u>: Condition on X_1.



$X_{n+1}=0 \iff$ all 3 sub-trees die out within n generations.
These have prob. q_n each.

$$P(X_{n+1}=0 \mid X_1=k) = q_n^{k}$$

so $$P(X_{n+1}=0 \mid X_0=1) = \sum P(\xi=k)\cdot q_n^{k}$$
$$= g(q_n)$$