2023-04-07

<u>Recall</u>: each indiv. has iid number of children, ξ_v

<u>Thm</u>: $P(\text{survival}) > 0 \iff \mathbb{E}\xi > 1$ (or $P(\xi = 1) = 1$)

<u>denote</u> $g(s) = \mathbb{E}s^{\xi} = \sum_{n=0}^{\infty} p(n) s^n$ is the prob gen. func. of ξ.

<u>Claim</u>: $P(X_{n+1} = 0) = g(P(X_n = 0))$

$X_0 = 1$ so $P(X_0 = 0) = 0$.

Let $q_n = P(X_n = 0)$ then $q_0 = 0$ $q_1 = g(0)$

$q_2 = g(q_1) = g(g(0))$

$q_n = g(g(\cdots g(0)\cdots))$

Extinction = $X_n = 0$ for some n

$$P(\text{extinction}) = \lim_{n\to\infty} q_n$$

e.g. $\xi \sim \text{Bin}(2, 2/3)$ then $\quad g(s) = \left(\frac{1}{3} + \frac{2}{3}s\right)^2$

$$q_1 = g(0) = \frac{1}{9}$$

$$q_2 = g(q_1) = \left(\frac{11}{27}\right)^2$$

<u>note</u>: q_n is increasing so $\lim q_n$ exists.

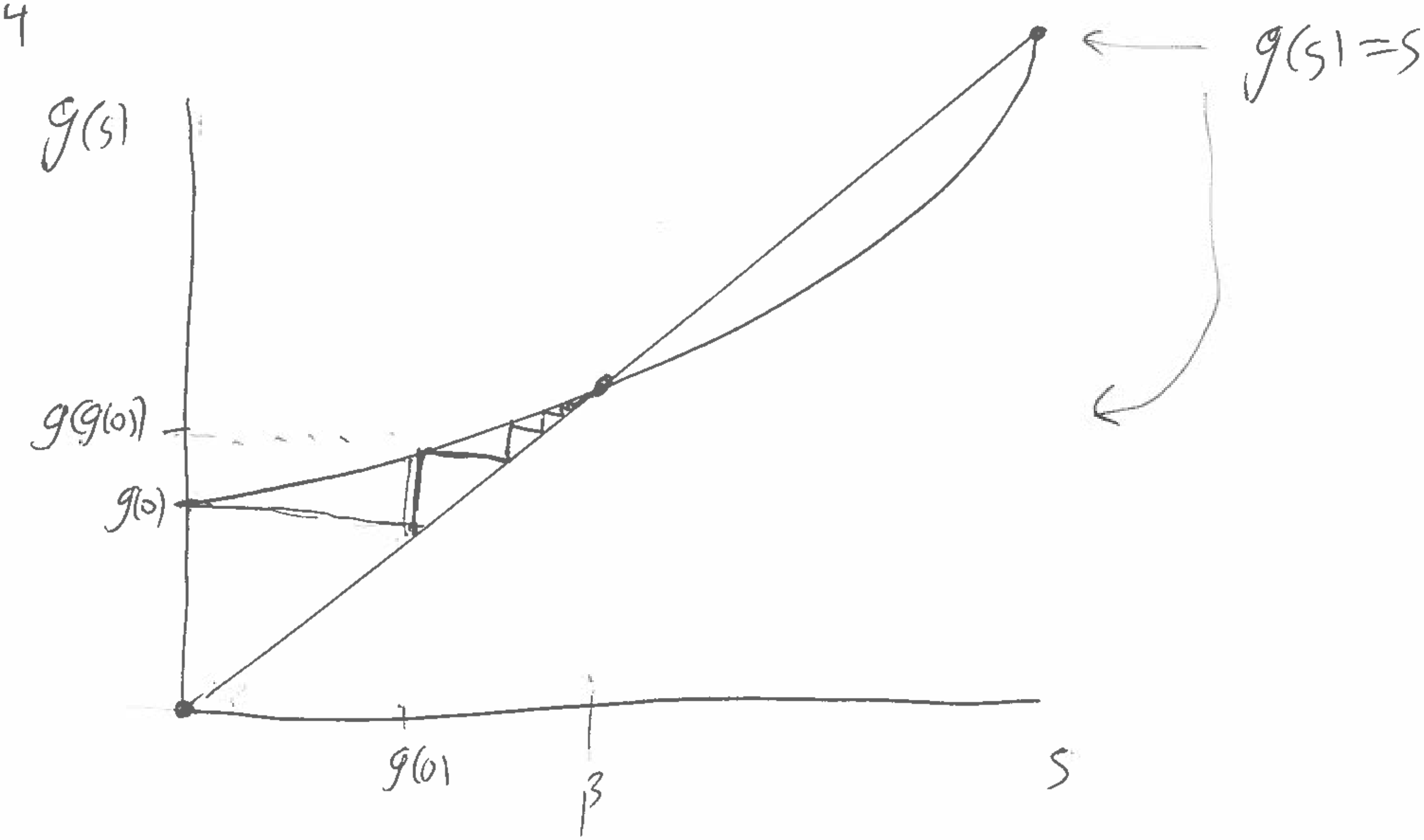
$$\begin{array}{ccc} q_{n+1} & = & g(q_n) \\ \downarrow & & \downarrow \\ q_\infty & = & g(q_\infty) \end{array} \qquad q_\infty = P(\text{extinction})$$

__Thm__: q_∞ is the smallest sol. in $[0,1]$ of $q = g(q)$

In $\mathrm{Bin}(2, \tfrac{2}{3})$: $\quad s = g(s) = \left(\frac{1}{3} + \frac{2}{3}s\right)^2 = \frac{4s^2+4s+1}{9}$

$4s^2 - 5s + 1 = 0 \implies s = 1$ or $\underline{s = \frac{1}{4}}$

so $q_\infty = \frac{1}{4}$



Proof idea: By induction we show that $q_n < \beta$ where $\beta =$ smallest sol. of $g(s) = s$

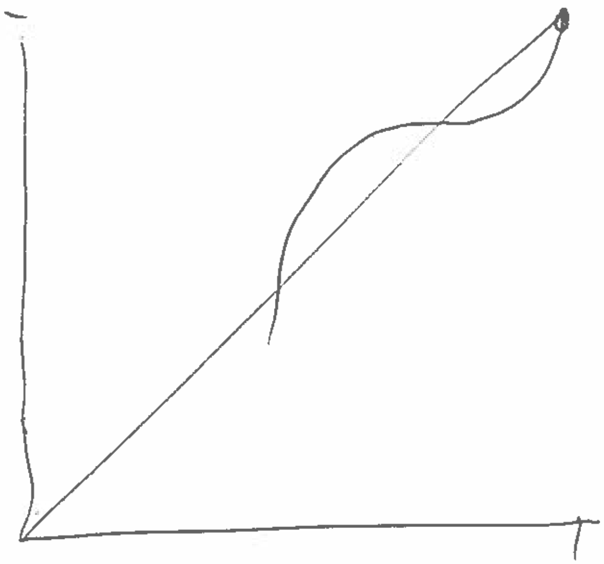
q_n increasing so has a limit q_∞

so $q_\infty \le \beta$. Since $q_\infty = g(q_\infty)$ we must have $q_\infty = \beta$. $\square$

Claim: $g(s) = s$ has a sol. at $s = 1$ and at most one other sol.

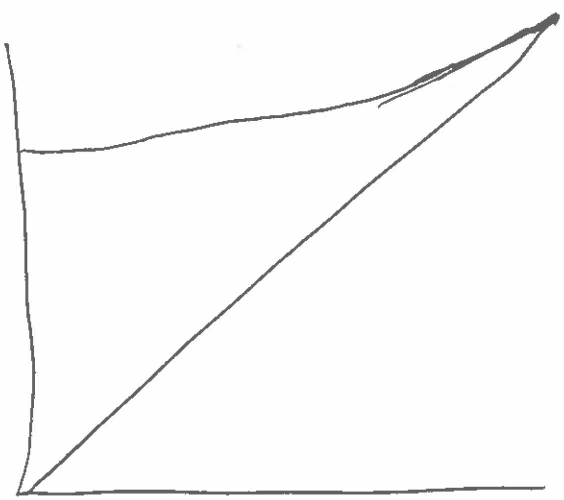
There is a second. sol. $\Longleftrightarrow g'(1) > 1$

Proof: g convex since $g'' \geqslant 0$



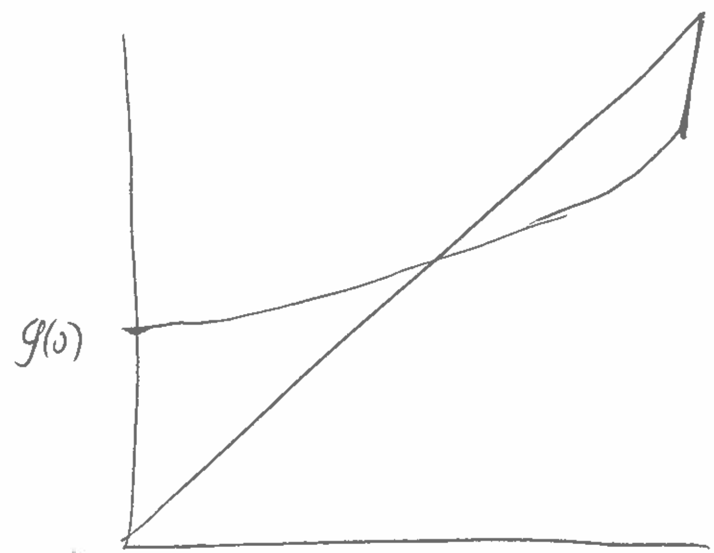
convexity rules out more sol. to $g(s)=s$.

if $g'(1) < 1$:



no sol. < 1

$g'(1) > 1$



must have a sol.

$$g'(1) = \frac{d}{ds}\left(\sum p(n)\, s^n\right)\Big|_{s=1} = \sum_n p(n)\cdot n\, s^{n-1}\Big|_{s=1} = \sum p(n)\cdot n = E\xi$$

Markov Chain Monte Carlo

<u>Idea</u>: given a complex distrib. π generate samples of π by running a M.C. with π the stat. dist.

There are <u>many</u> M.C.s for any π.

<u>Want</u>: Fast mixing: $P(X_n = i) \xrightarrow[n\to\infty]{} \pi_i$ quickly.

Easy to simulate.

<u>note</u>: one way to build a M.C. is with reversibility

given π pick transitions s.t.

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \text{for all } i,j$$

Can take arbitrary P_{ii} to get stochastic P.

e.g. <u>Magnetism + Ising</u>:

Each atom can point up or down only

$$\begin{array}{cccc} + & - & - & + \\ - & - & + & + \\ - & + & + & + \end{array} \qquad \sigma = (\sigma_x)_{x \in V}$$

$$\sigma_x \in \{\pm 1\}$$

energy of a state:

$$H(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v$$ sum over neighbouring pairs.

<u>Boltzmann</u>: $$P(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)}$$

$\beta = \frac{1}{T}$ inverse temp.

Z = partition func. (normalizing)

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high $\beta \Rightarrow$ concentrate on low energy states

low $\beta \Rightarrow$ more uniform.