Mathematics 318 — Sample midterm hints

1. Let X have c.d.f. $F(t) = \begin{cases} 0 & t < 0, \\ \sin(t/2) & t \in [0, \pi], \\ 1 & t > \pi. \end{cases}$
   (a) What is $E[X]$?
   (b) What are the CDF and PDF of $Y = \sqrt{X}$?

   Solution:
   (a) the pdf i F', so $E[X] = \int_0^\pi xF'(x)dx$. Calculate the integral. (Or use $E[X] = \int_0^\infty 1 - F(x)dx$.)
   (b) $P(Y \le t) = P(X \le t^2) = F(t^2)$ gives the cdf of Y. differentiate for the pdf.

2. Let X, Y be random variables with joint p.d.f. $f(x, y) = x + y$ on the square $[0, 1] \times [0, 1]$ and 0 outside.
   (a) Are X and Y independent?
   (b) What is the marginal distribution of X?
   (c) What is the Covariance $\mathrm{Cov}(X, Y)$?

   Solution:
   (a) $x + y$ is not a product of $f_X(x)f_Y(y)$.
   (b) Integrate $\int_0^1 f(x, y)dy$.
   (c) Calculate $E[XY] = \iint xyf(x, y)dxdy$ and similarly $E[X]$ and $E[y]$ to find the covariance.

3. Let X, Y be uniform on the square $[0, 2] \times [0, 2]$. Find the distribution of the ratio $Z = X/Y$.

   Solution: $P(Z \le t)$ is the probability that (X, Y) is in the square and above the line $x = ty$. This is the ratio of the area of the domain and the whole square. This comes to $t/2$ for $t \in [0, 1]$ and $1 - 1/2t$ for $t \in [1, \infty)$.

4. Let (X, Y) be the result of rolling two (6-sided) dice.
   (a) What is the characteristic function of X?
   (b) What is the characteristic function of $X + Y$?
   (c) Are the events A, B, C independent, where

$$A = \{X \text{ is even}\}, \qquad B = \{y \text{ is even}\}, \qquad C = \{X + Y = 5\}.$$

(d) Which pairs out of A, B, C are independent?

Solution:

(a) $\phi(t) = \sum_{k=1}^{6} e^{ikt}$.
(b) $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) = \phi_X(t)^2$.
(c) No. $P(A) = P(B) = 1/2$ and $P(C) > 0$ but $P(A \cap B \cap C) = 0$.
(d) Any two are independent. (justify!)

5. An urn contains 4 red, 6 green, and 10 blue balls.
   (a) If three balls are drawn **with** replacement, what is the probability that the balls have three different colours?
   (b) If three balls are drawn **without** replacement, what is the probability that they are all the same colour?
   (c) If three balls are drawn **without** replacement, what is the probability that they are all blue conditioned on the event that they all have the same colour?

Solution: Explain these:

(a)
$$P = 3!\frac{4}{20}\frac{6}{20}\frac{10}{20}$$

(b)
$$P = \frac{\binom{4}{3} + \binom{6}{3} + \binom{10}{3}}{\binom{20}{3}}$$

(c)
$$P = \frac{\binom{10}{3}}{\binom{4}{3} + \binom{6}{3} + \binom{10}{3}}$$