Problem 1. What is a game? Give examples of games with and without randomness, and with or without possible ties (all 4 combinations).

Solution. There are many possible definitions. Any interaction between players where they make choices which lead to some outcomes can be considered a game. Solitaire games are possible, and are sometimes interesting if there is an element of chance or skill involved.

Many examples exist. Chess and Go are not random, Poker and Backgammon are. Other examples: Snooker (the set of actions is determined by the player’s skill), The tragedy of the commons, The prisoner’s dilemma.

Problem 2. Alice and Bob play tic-tac-toe. Alice goes first. Consider the following strategy $S$:

- If I can win right now, play to win.
- If not, and the opponent is about to win, block them.
- Otherwise, play in the center if possible.
- Otherwise, play in a corner if possible (first available, top to bottom, left to right).
- Otherwise play in a side square (same order of preference).

Questions:
(a) If both players use strategy $S$, what is the outcome of the game? Draw the resulting board.
(b) If only Bob (as player 2) uses strategy $S$, show that Alice has a strategy that wins.

Solution.
(a) Using $X_i$ for Allice's moves and $O_i$ for Bob, the result is a draw:

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    O2   O8   X3
X5   X1   O6
O4   X9   X7
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(b) Alice starts at a corner. Bob replies in the centre. Alice moves in the opposite corner. Bob takes a third corner. Alice takes the last corner and wins in her next move:

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  X1   O4
X5   O2
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Problem 3. Recall the game of CHOMP from class (see Chapter 1 of Karlin+Peres).
(a) How many possible positions might come up during the game starting with a $3 \times 3$ board? Include the starting and final position.
(b) [bonus] How many possible position in the game started with an $n \times m$ board?

Solution.
(a) There are 20 (or 19 excluding the empty board.)
(b) There are \( \binom{n+m}{n} \) positions. (One less if excluding the empty board.) The contour of a position is a path from $(0, m)$ to $(n, 0)$ using $n$ right steps and $m$ down steps. The $n+m$ steps can be in any order, and there are \( \binom{n+m}{n} \) possible orders.

Problem 4. Alice and Bob play CHOMP, starting with a $2 \times 4$ board. Alice goes first.
(a) Alice’s strategy is to always take a single square from the top row if possible, or from the bottom if the top row is all gone. Is this a winning strategy? If not, give a strategy for Bob that wins against this.
(b) What if Bob goes first? (with Alice still using the same strategy.)
(c) What happens if both players use this strategy?

[bonus] Describe a winning strategy for the first player.
Solution.

(a) This is not a winning strategy. After Alice moves, Bob can take the entire top row. Alice takes one square, and Bob takes 2 leaving only the final square for Alice to lose.
(b) This is also not winning if Bob goes first. Bob takes the whole top row and proceeds as before.
(c) If both use this strategy each takes a single squares, and after the 7th move (by Player 1), there is only the losing square left, so Player 1 wins.