Mathematical Game theory

Assignment 2 solutions

Problem 1. Find the set of P-positions for normal winning condition with the following subtraction sets. In all cases, prove your claims.

$$(a)\ \{1,2,5\} \qquad (b)\ \{1,3,5\} \qquad (c)\ \{1,4,5\} \qquad (d)\ \{3,4,5\}.$$

Solution. In all cases, to prove that the set of P-positions is P and the set of N-positions is $N = P^c$ involves showing that:

- every $x \in N$ has a follower in P, and
- every $x \in P$ has no followers in N (possibly no followers at all).

(a) The P-positions are multiples of 3. If $n = 3k$ then the followers $3k-1, 3k-2, 3k-5$ are not multiples of 3, so are N-positions. If $n = 3k+1$ then taking 1 chip gives a follower which is a multiple of 3. If $n = 3k+2$ then taking 2 chips (or 5) gives a follower which is a multiple of 3.

(b) The P-positions are the even numbers. If $n = 2k$ then the followers $2k-1, 2k-3, 2k-5$ are all odd, so are N-positions. If $n = 2k+1$ then taking 1 chip gives an even follower.

(c) The P-positions are $P = \{8n\} \cup \{8n+2\}$. If n has remainder 0 modulo 8, the followers have remainders 3,4,7 modulo 8. If n has remainder 2 modulo 8, the followers have remainders 1,5,6 modulo 8. In both cases, all followers are in $N = P^c$. If $n = 8k+1$ or $8k+3$, taking 1 chip gives a P-position. If $n = 8k+4$ or $8k+6$, taking 4 chips gives a P-position. If $n = 8k+5$ or $8k+7$, taking 5 chips gives a P-position.

(d) The P positions are all n which are 0,1, or 2 $\pmod 8$. From any such n there are no moves to another such n, and from any other n taking 3 or 5 chips will leave such an n.

See below for sample python code to discover these results.

Problem 2. Consider a variation of the subtraction game with set $A = \{1,2,3,4\}$. The new rule is that instead of taking coins from the pile, a player may instead return 1,2,3, or 4 previously taken coins to the pile. Players can only use coins they previously took from the pile.

(a) Show that this game is not progressicely bounded.

(b) Prove that starting with 128 coins, player 1 can still guarantee a win no matter what player 2 does.

Solution.

(a) It is possible that each player alternates taking a chip and returning it, so the game can last any number of turns, even if started from only 2 chips.

(b) Player 1 will take 3, and afterwards always leave a multiple of 5 (as in the usual subtraction game), and **never return any chips**. Eventually, there will be no chips in the pile. Player 2 might return some, and player 1 again takes all. Eventually, player 2 will run out of chips to return and lose.

Problem 3. Consider the subtraction game with subtraction set the set of squares $A = \{1,4,9,16,\dots\}$. Prove that there are infinitely many P-positions. (Hint: use proof by contradiction.)

Solution. Assume for a contradiction that there is a largest P-position, and call it m. Let n be such that there are no squares in $[n, n+m]$. Then every legal move from $n+m$ leaves more than m coins in the pile. So all followers of $n+m$ are N-positions, and therefore $n+m$ is a P-position. This contradicts our assumption that the larges P-position is m.

Note: one way to find such n is $n = m^2+1$, since the next larget square is $(m+1)^2$. See also the class notes for a double counting proof.

Problem 4. There are two boxes. Initially, one box contains m chips and the other contains n chips. Such a position is denoted by (m, n), where $m > 0$ and $n > 0$. The two players alternate moving. A move consists of emptying one of the boxes, and dividing the contents of the other between the two boxes with *at least one chip* in each box. There is a unique terminal position, namely $(1, 1)$. Consider Misere rules: last player to move loses. Show that the only P-positions are of the form

$$(3k-1, 1), (1, 3k-1), \text{ or } (3k-1, 3l-1),$$

where $k, l > 0$ are arbitrary natural numbers.

Solution. Let P be the claimed set of P-position. First, we show that if $(m, n) \in P$, then every follower is in $N = P^c$. If $m = 1$ or $n = 1$, then the box with a single stone must be emptied. Therefore the number of stones that is divided between the boxes is of the form $3k - 1$, and the follower is (m', n') with $m' + n' = 3k - 1$. However in every position in P the total number of chips is either a multiple of 3 or 1 modulo 3, and never -1.

Second, we show that every position in N has a follower in P. Suppose a position (m, n) is in N. If m is divisible by 3, move to $(1, m-1) \in P$, and similarly if n is divisible by 3. If m is 1 modulo 3 and $m > 1$, then move to $(2, m-2) \in P$ (and similarly with $n - 1$ divisible by 3).

Finally, we note that the terminal position is in N, since this is misere.

Bonus Problem 5. Consider the subtraction game with set $\{1, n\}$. Find (with proof) the period of the set of P-positions as a function of n.

Solution. If n is odd, the parity of the pile changes every step. Since the only terminal position is 0, the number of moves is the same parity as the initial n. Therefore Player 1 will win for odd n and Player 2 will win for even n.

If $n = 2k$ is even, then the period is $n + 1$, and the P-positions are

$$P = \{x : x \mod (n+1) \in \{0, 2, 4, \ldots, 2k-2\}\}.$$

For example, for $n = 8$ the P-positions are numbers that are 0,2,4, or 6 $\pmod 9$. To see this, note first that taking 1 from such a P-position leaves a result that is in $\{-1, 1, 3, \ldots, 2k-3\} \pmod{()n+1}$, and $-1 \equiv n$, so these followers are N-positions. Taking n from a pile is equivalent to adding 1 mod $(n+1)$, and so taking n from a P-position leaves an N-position.

Second, suppose x is an N-position. If $x \mod (n+1) \in \{1, 3, \ldots, 2k-1\}$ the taking 1 from x will leave a P-position. This leaves $x \equiv n \pmod{()n+1}$, in which case taking n will leave a P-position.

Bonus Problem 6. Write python code for a function `sub(n,A)` which returns 0 if n is a P-position for subtraction with set A and 1 otherwise. Use this to calculate the eventual period for the sets $\{2, 5, 11\}$ and $\{3, 6, 17\}$.

Solution. Possible python code to

```
from functools import *

@cache
def sub(n,A):
    """return True if n is a P-position, and False if N-position."""
    for a in A:
        if a<=n and sub(n-a,A): #we found a P-pos follower
            return False
    return True

print([sub(n,(1,2,5)) for n in range(40)])
```

The decorator `@cache` instructs python to remember previously computed values of the function instead of re-computing them. With this, the formula is used once for each possible board configuration. Without it, the program will essentially go over all possible games, and becomes much much slower.