Problem 1. Consider the subtraction game.
(a) Find the set of P-positions for normal winning condition with subtraction set \{1,2,5\}.
(b) Find the set of P-positions for normal winning condition with subtraction set \{1,4,5\}.
(c) Find the set of P-positions for normal winning condition with subtraction set \{3,4,5\}.
Justify your answers! (always)

Solution.
(a) The P-positions are multiples of 3. If \(n = 3k\) then the followers \(3k - 1, 3k - 2, 3k - 5\) are not multiples of 3, so are N-positions. If \(n = 3k + 1\) then taking 1 chip gives a follower which is a multiple of 3. If \(n = 3k + 2\) then taking 2 chips (or 5) gives a follower which is a multiple of 3.
(b) The P-positions are \(P = \{8n\} \cup \{8n + 2\}\). If \(n\) has remainder 0 modulo 8, the followers have remainders 3,4,7 modulo 8. If \(n\) has remainder 2 modulo 8, the followers have remainders 1,5,6 modulo 8. In both cases, all followers are in \(P^c\). If \(n = 8k + 1\) or \(8k + 3\), taking 1 chip gives a P-position. If \(n = 8k + 4\) or \(8k + 6\), taking 4 chips gives a P-position. If \(n = 8k + 5\) or \(8k + 7\), taking 5 chips gives a P-position.
(c) The P positions are all \(n\) which are 0,1, or 2 \pmod{8}. From any such \(n\) there are no moves to another such \(n\), and from any other \(n\) taking 3 or 5 chips will leave such an \(n\).

Problem 2. Consider a variation of the subtraction game with set \(A = \{1,2,3,4\}\). The new rule is that instead of taking coins from the pile, a player may instead return 1,2,3, or 4 previously taken coins to the pile (but only the coins that came from the pile).
(a) Prove that this game is not progressicely bounded.
(b) Prove that starting with 128 chips, player 1 can still guarantee a win no matter what player 2 does.

Solution.
(a) It is possible that each player alternates taking a chip and returning it, so the game can last any number of turns, even if started from only 2 chips.
(b) Player 1 will take 3, and afterwards always leave a multiple of 5 (as in the usual subtraction game), and never return any chips. Eventually, there will be no chips in the pile. Player 2 might return some, and player 1 again takes all. Eventually, player 2 will run out of chips to return and lose.

Problem 3. Consider the subtraction game with subtraction set the set of squares \(A = \{1,4,9,16,\ldots\}\). Prove that there are infinitely many P-positions. (Hint: use proof by contradiction.)

Solution. Assume for a contradiction that there is a largest P-position, and call it \(m\). Let \(n\) be such that there are no squares in \([n, n+m]\). Then every legal move from \(n+m\) leaves more than \(m\) coins in the pile. So all followers of \(n + m\) are N-positions, and therefore \(n + m\) is a P-position. This contradicts our assumption that the larges P-position is \(m\).

Note: one way to find such \(n\) is \(n = m^2 + 1\), since the next largest square is \((m + 1)^2\). See also the class notes for a double counting proof.

Problem 4. There are two boxes. Initially, one box contains \(m\) chips and the other contains \(n\) chips. Such a position is denoted by \((m,n)\), where \(m > 0\) and \(n > 0\). The two players alternate moving. A move consists of emptying one of the boxes, and dividing the contents of the other between the two boxes with at least one chip in each box. There is a unique terminal position, namely \((1,1)\). Consider Misere rules: last player to move loses. Show that the only P-positions are of the form \((3k - 1, 1), (1, 3k - 1), \) or \((3k - 1, 3l - 1)\), where \(k, l > 0\) are arbitrary natural numbers.
Solution. Let $P$ be the claimed set of $P$-position. First, we show that if $(m, n) \in P$, then every follower is in $N = P^c$. If $m = 1$ or $n = 1$, then the box with a single stone must be emptied. Therefore the number of stones that is divided between the boxes is of the form $3k - 1$, and the follower is $(m', n')$ with $m' + n' = 3k - 1$. However in every position in $P$ the total number of chips is either a multiple of 3 or 1 modulo 3, and never $-1$.

Second, we show that every position in $N$ has a follower in $P$. Suppose a position $(m, n)$ is in $N$. If $m$ is divisible by 3, move to $(1, m - 1) \in P$, and similarly if $n$ is divisible by 3. If $m$ is 1 modulo 3 and $m > 1$, then move to $(2, m - 2) \in P$ (and similarly with $n - 1$ divisible by 3).

Finally, we note that the terminal position is in $N$, since this is misere.

Python code for problem 1.

Here is sample python code to generate the sequences in problem 1:

```python
@memoize
def sub(n,A):
    """return True if n is a P-position, and False if N-position."""
    for a in A:
        if a<=n and sub(n-a,A): #we found a P-pos. follower
            return False
    return True

print([sub(n,(1,2,5)) for n in range(40)])
```

The decorator `@memoize` instructs python to remember previously computed values of the function instead of re-computing them. With this, the formula is used once for each possible board configuration. Without it, the program will essentially go over all possible games, and becomes much slower. Include the following code.

```python
def memoize(f):
    "Memoization decorator for functions taking one or more arguments."
    class memodict(dict):
        def __init__(self, f):
            self.f = f
        def __call__(self, *args):
            return self[args]
        def __missing__(self, key):
            ret = self[key] = self.f(*key)
            return ret
    return memodict(f)
```