Mathematical Game theory

Assignment 3 solutions

Problem 1. Find the Nim-sum of all numbers from 1 to $2^n - 1$, where $n > 1$ is a natural number. More precisely, compute

$$1 \oplus 2 \oplus \cdots \oplus (2^n - 1).$$

Hint: Show that there are exactly 2^{n-1} numbers with 1 as the kth digit of their binary expansion for each $k = 1, 2, \ldots, n$.

Solution. Each sequence of digits $(a_0, \ldots, a_n)$ with $a_i \in \{0, 1\}$ corresponds to a number $a = \sum a_i 2^i$, and this gives all numbers from 0 to $2^n - 1$. If we want numbers where $a_k = 1$, there are exactly 2^{n-1} possibilities for the other digits. Therefore the nim-sum $0 \oplus 1 \oplus \cdots \oplus (2^n - 1)$ has 2^{n-1} ones in each position. Since 2^{n-1} is even, the required sum is 0. Note that this includes 0 in the sum, but adding 0 makes no difference (as with regular addition).

Alternative solution. We can add consecutive pairs in the sum. For $n = 2$ this would be

$$0 \oplus 1 \oplus 2 \oplus 3 = (0 \oplus 1) \oplus (2 \oplus 3) = 1 \oplus 1 = 0.$$

In general, pair $2k$ with $2k + 1$. Since $2k \oplus (2k + 1) = 1$, we get

$$S = 1 \oplus (2 \oplus 3) \oplus (4 \oplus 5) \oplus \cdots = 1 \oplus 1 \oplus 1 \oplus \cdots = 0,$$

Since there are 2^{n-1} 1's (an even number).

Problem 2. Find a formula for the Sprague-Grundy value of a pile of size n in the following subtraction games:

(a) Subtraction set is all multiples of 5.
(b) Subtraction set is $\{2, 3, 7\}$.
(c) Subtraction set is the set of primes: find $g(n)$ for $n \leq 10$.

Solution.

(a) We show by induction that $g(n) = [n/5]$, where $[x]$ is the integer part of x. For $x < 5$ there are no followers, so indeed $g(x) = 0$. For larger x the followers are $\{x - 5, x - 10, \ldots\}$. If $x = 5n$, then the followers are $\{0, 5, 10, \ldots, 5n - 5\}$. By the induction hypothesis, these ave g values $\{0, 1, \ldots, n - 1\}$ so the mex is n. Similarly, if $x = 5n + i$ for some $i < 5$, then the followers are $\{i, i + 5, \ldots, i + 5n - 5\}$. In all these cases the g-values of the followers are also $\{0, 1, \ldots, n - 1\}$ so the mex is n.

(b) Computing the first few cases shows that $g(n)$ is periodic, with repeating values $[0, 0, 1, 1, 2]$, so

$$g(x) = \begin{cases} 0 & (x \mod 5) \in \{0, 1\}, \\ 1 & (x \mod 5) \in \{2, 3\}, \\ 2 & (x \mod 5) = 4. \end{cases}$$

We check this by hand for $x < 7$. For larger x, the followers are $\{x - 2, x - 3, x - 7\}$, which are equivalent to $x - 2$ and $x - 3$ modulo 5. Checking the 5 possibilities for $x \mod 5$ completes the proof. For example, if $x \equiv 1 \pmod 5$, then the followers are 3 and 4 modulo 5, and so have g-values $1, 2$ and the mex is 0, as required.

(c) The first few values (for $n = 0 \ldots 19$) are

$$0, 0, 1, 1, 2, 2, 3, 3, 4, 0, 0, 1, 1, 2, 2, 3, 3, 4, 0, 0$$

For example, the followers of 10 are $8, 7, 5, 3$ with values 4,3,2,1 respectively, so $g(10) = \text{mex}(\{4, 3, 2, 1\}) = 0$. Code to do this by computer:

```
from functools import *

def mex(S):
    A = set(S)
    for i in range(1+len(A)):
        if i not in A: return(i)

primes = [2,3,5,7]

@cache
def g(n,A):
    if n==0: return 0
    return mex(list(g(n-x) for x in A if x<=n))

print([g(n,primes) for n in range(20)])
```

Problem 3. A graph is a set of vertices, and a set of edges, where each edge is a pair of vertices. A graph is drawn as dots for vertices with a line for each edge. Consider the following game: A move is to pick a vertex which has at least one edge attached to it, and delete that vertex and all edges containing it. A vertex with no remaining edges cannot be deleted.

(a) Find the Sprague-Grundy value of the star S_n with n leaves n.
(b) Show that the Sprague-Grundy value of a cycle C_n is either 0 or 1.
(c) Find the Sprague-Grundy value of the path P_n of length n for $n \leq 10$.
(d) Show that if n is even then P_n is an N-position (thus $g(P_n) \neq 0$).

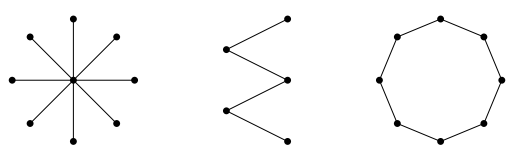

Figure 1: The star S_8, path P_4 and cycle C_8.

Solution.

(a) From S_n taking the centre vertex leaves a move with value 0. Taking a leaf leaves S_{n-1}. Therefore $g(S_0) = 0$ and $g(S_n) = \text{mex}\{0, g(S_{n-1}\}$, which implies values alternate between 1 and 2.
(b) From C_n the only move is to P_{n-2}. Therefore $g(C_n) = 1$ if $g(P_{n-2}) = 0$, and $g(C_n) = 0$ otherwise. (In general, if a position x has m followers then $g(x) \leq m$.)
(c) From P_n we can get either P_{n-1} or $P_i \oplus P_{n-2-i}$. We find by hand that the first few values for $n = 0 \ldots, 10$ are $(0, 1, 2, 0, 1, 2, 3, 1, 2, 3, 4)$. (See code below.) There is no clear pattern here. The only P-positions up to 200000 (found by computer) are $\{0, 3, 11, 19, 29, 45, 71, 97, 123, 149, 175, 313, 407\}$ and it is believed there are no others.
(d) If n is even, then P_n has the follower two copies of $P_{n/2-1}$ which cancel out. Thus $g(P_n) \neq 0$ for even n, and so $g(P_n) \neq 0$ for n even.
There is also an explicit winning strategy for second player on the cycle of even length: always take the vertex opposite your opponent's move.

Problem 4. Consider a subtraction game where the valid moves are either to take a single chip from the pile, or to take exactly half the pile (only possible if the size is even).

(a) Prove that every odd n with $n \geq 5$ is a P-position.

(b) Find (with proof) $g(10^{100})$. (Hint: relate $g(2n)$ to $g(n)$.)

Solution.

(a) We find by hand that the first few values for $n \leq 8$ are $0, 1, 0, 1, 2, 0, 2, 0, 1$. By induction we show that, for odd $n \geq 5$, we have $g(n) = 0$, and for even n we have that $g(n)$ is either 1 or 2. The odd case holds since odd n has a single even follower $n - 1$, with value in $\{1, 2\}$. In the even case n has two followers with values $g(n - 1) = 0$ and $g(n/2) \in \{0, 1, 2\}$, so $g(n) \in \{1, 2\}$.

(b) The key idea is that if n is even and $g(n) \in \{1, 2\}$ then $g(2n)$ is the other of $\{1, 2\}$. Thus $g(2^k n)$ alternates on $\{1, 2\}$. If n is odd then $g(2n) = \text{mex}\{g(2n - 1), g(n)\} = \text{mex}\{0, 0\} = 1$. Thus the values along the sequence $2^k 5^{100}$ are $0, 1, 2, 1, 2, \ldots$, and for $k = 100$ we get $g(10^{100}) = 2$. (More generally, if k is odd, then $g(k2^n)$ is 1 if n is odd and 2 if n is even

Problem 5. There are two piles of chips. A valid move is to take any number of chips from one of the piles (at least one, as in NIM) **or** to take a single chip from both piles. Find a formula for the Sprague-Grundy value $g(n, 1)$.

Solution. If one of the piles is empty, this is just NIM, so $g(n, 0) = n \oplus 0 = n$. From $(n, 1)$ the possible moves are to $(m, 1)$ with $m < n$, or to $(n, 0)$ or to $(n - 1, 0)$. We can find by hand that the first few values of $g(n, 1)$ are $1, 2, 0, 4, 5, 3, 7, 8, 6, \ldots$ Split these to triplets to see a clear pattern. The general pattern is

$$g(n, 1) = \begin{cases} n + 1 & n \equiv 0 \text{ or } n \equiv 1 \pmod 3, \\ n - 2 & n \equiv 2 \pmod 3. \end{cases}$$

To prove this, we check the 3 cases. For example, if $n = 3k$ than all $i < 3k$ are values of $g(m, 1)$ for $m < n$, and $g(3k, 0) = 3k$, so the mex is $3k + 1$. The two other cases are similar.

Bonus Problem 6. For the game from Problem 5:

(a) Find a general formula for $g(x, y)$.

(b) Prove your the formula.

Solution. The formula: If $x, y \in \{0, 1, 2\}$ then $g(x, y) = (x + y) \mod 3$. For larger values it turns out that for any $x, y \leq 3$ and any $a, b \in \mathbb{N}$:

$$g(3a + x, 3b + y) = g(x, y) + 3(a \oplus b).$$

It is possible to prove this by checking many cases. A shorter proof uses a mixed base 2-3 representation. Write $x = a_0 + 3\sum a_i 2^i$, and $y = b_0 + 3\sum b_i 2^i$, with $a_0, b_0 \in \{0, 1, 2\}$ and the others in $\{0, 1\}$. Then the claim is that

$$g(x, y) = (a_0 + b_0 \mod 3) + \sum (a_i \oplus b_i) 2^i.$$

Thus digits of x, y are added without carry. To prove this is the Sprague-Grundy value, we need to show two things:

- If (x', y') is a follower of (x, y) then $g(x', y') \neq g(x, y)$,
- For every $s < g(x, y)$ there is a follower with $g(x', y') = s$.

The first is obvious: changing some digits changes the sum. The second is similar to the case of NIM. If s differs from $g(x, y)$ in some digit except the least, then the in first place this occurs s has digit 0 and $g(x, y)$ has a 1 (since $s < g(x, y)$. This means we can remove chips from one pile to get to value s.

The remaining case is when all but the last digit of s agree with $g(x, y)$. Let the last digits of s and $g(x, y)$ be s_0, c_0.

- If $s_0 = 0, c_0 = 1$ then one of a_0, b_0 is non-zero and we take one chip from that pile.
- If $s_0 = 1, c_0 = 2$ then also one of a_0, b_0 is non-zero and we take one chip from that pile.
- If $s_0 = 0, c_0 = 2$ and one of a_0, b_0 is 2, we take two chips from that pile.
- The remaining case is $s_0 = 0, c_0 = 2, a_0 = b_0 = 1$. Here we take one chip from each pile.

Bonus Problem 7. Write a python program to compute the Sprague-Grundy of a chomp board. After executing your file, there should be a function `chomp(A)` which takes a tuple A and returns the value of that board. For example, the 5×5 board is `A=(5,5,5,5,5)` and `chomp(A)` should return 6. If 3 squares are missing from the top row, this is described as `(5,5,5,5,2)`. The lengths of rows are given from the bottom up.

Solutions to this problem must be submitted separately on canvas. I can only accept python code. A 5×5 board should not take more than a second to calculate. As a test, `\chomp((3,3))` and `\chomp((2,2,2))` should both return 4 (as computed in class).

Solution. My solution:

```
from functools import cache

def moves(B):  # possible moves from position B, given as a tuple.
    L = len(B)
    for i in range(L):
        for j in range(B[i]):
            if i==j==0: continue
            yield tuple((B[k] if k<i else min(B[k],j)) for k in range(L))

def mex(S):
    A = set(S)
    for i in range(1+len(A)):
        if i not in A: return(i)

# cache makes the recursion much faster for larger boards
@cache
def chomp(x):
    return mex(chomp(y) for y in moves(x))
```