Problem 1. Find the Nim-sum of all numbers from 1 to $2^n - 1$, where $n > 1$ is a natural number. More precisely, compute

$$1 \oplus 2 \oplus \cdots \oplus (2^n - 1).$$

Hint: Show that there are exactly $2^{n-1}$ numbers with 1 as the $k$th digit of their binary expansion for each $k = 1, 2, \ldots, n$.

**Solution.** Each sequence of digits $(a_0, \ldots, a_n)$ with $a_i \in \{0, 1\}$ corresponds to a number $a = \sum a_i 2^i$, and this gives all numbers from 0 to $2^n - 1$. If we want numbers where $a_k = 1$, there are exactly $2^{n-1}$ possibilities for the other digits. Therefore the nim-sum $0 \oplus 1 \oplus \cdots \oplus (2^n - 1)$ has $2^{n-1}$ ones in each position. Since $2^{n-1}$ is even, the required sum is 0.

**Alternative solution.** We can add pairs in the sum. Since $2^n \oplus (2^n + 1) = 1$, we get

$$S = 1 \oplus (2 \oplus 3) \oplus (4 \oplus 5) \cdots = 1 \oplus 1 \oplus 1 \cdots = 0,$$

Since there are $2^{n-1}$ 1’s.

Problem 2. Find a formula for the Sprague-Grundy value of a pile of size $n$ in the following subtraction games:

(a) Subtraction set is all multiples of 5.
(b) Subtraction set is $\{2, 3, 7\}$.

**Solution.**

(a) We show by induction that $g(n) = \lfloor n/5 \rfloor$, where $\lfloor x \rfloor$ is the integer part of $x$. For $x < 5$ there are no followers, so indeed $g(x) = 0$. For larger $x$ the followers are $\{x - 5, x - 10, \ldots\}$. If $x = 5n$, then the followers are $\{0, 5, 10, \ldots, 5n - 5\}$. By the induction hypothesis, these have $g$ values $\{0, 1, \ldots, n - 1\}$ so the mex is $n$. Similarly, if $x = 5n + i$ for some $i < 5$, then the followers are $\{i, i + 5, \ldots, i + 5n - 5\}$. In all these cases the $g$-values of the followers are also $\{0, 1, \ldots, n - 1\}$ so the mex is $n$.

(b) Computing the first few cases shows that $g(n)$ is periodic, with repeating values $\{0, 0, 1, 1, 2\}$, so

$$g(x) = \begin{cases} 
0 & (x \mod 5) \in \{0, 1\}, \\
1 & (x \mod 5) \in \{2, 3\}, \\
2 & (x \mod 5) = 4.
\end{cases}$$

We check this by hand for $x < 7$. For larger $x$, the followers are $\{x - 2, x - 3, x - 7\}$, which are equivalent to $x - 2$ and $x - 3$ modulo 5. Checking the 5 possibilities for $x \mod 5$ completes the proof. For example, if $x \equiv 1 \pmod{5}$, then the followers are 3 and 4 modulo 5, and so have $g$-values 1, 2 and the mex is 0, as required.

Problem 3. A graph is a set of vertices, and a set of edges, where each edge is a pair of vertices. A graph is drawn as dots for vertices with a line for each edge. Consider the following game: A move is to pick a vertex which has at least one edge attached to it, and delete that vertex and all edges containing it. A vertex with no remaining edges cannot be deleted.

(a) Find the Sprague-Grundy value of the star $S_n$ with $n$ leaves $n$.
(b) Find the Sprague-Grundy value of the path $P_n$ of length $n$ for $n \leq 10$. Note any patterns you identify.
(c) Show that the Sprague-Grundy value of a cycle $C_n$ of even length $n \geq 4$ is 0 and for odd $n \geq 3$ is either 0 or 1.
Solution.
(a) From $S_n$ taking the centre vertex leaves a move with value 0. Taking a leaf leaves $S_{n-1}$. Therefore $g(S_n) = 0$ and $g(S_n) = \text{mex}(0, g(S_{n-1}))$, which implies values alternate between 1 and 2.
(b) From $P_n$ we can get either $P_{n-1}$ or $P_1 \oplus P_{n-2}$. We find the first few values are $(0, 1, 2, 0, 1, 2, 3, 1, 2, 3, 4)$. There is no clear pattern. The only P-positions up to 200000 (found by computer) are $(0, 3, 11, 19, 29, 45, 71, 97, 123, 149, 175, \ldots)$ and it is believed there are no others.
(c) From $C_n$ the only move is to $P_{n-2}$. Thus $g(C_n)$ is either 0 or 1, depending on whether $g(P_{n-2})$ is positive or 0. If $n$ is even, then $P_n$ has the follower two copies of $P_{n/2-1}$ which cancel out. Thus $g(P_n) \neq 0$ for even $n$, and so $g(C_n) = 0$ for $n$ even.
There is also an explicit winning strategy for second player on the cycle of even length: always take the vertex opposite your opponent’s move.

Problem 4. Consider a subtraction game where the valid moves are either to take a single chip from the pile, or to take exactly half the pile (only possible if the size is even).
(a) Prove that every odd $n$ with $n \geq 5$ is a P-position.
(b) Find (with proof) $g(10100)$.

Solution.
(a) We find by hand that the first few values for $n \leq 4$ are 0, 1, 0, 1, 2. By induction we show that, for odd $n \geq 5$, we have $g(n) = 0$, and for even $n$ either 1 or 2 if $n$. This is since odd $n$ has a single even follower $n - 1$, and even $n$ has two followers with values $g(n-1) = 0$ and $g(n/2)$.
(b) The key idea is that if $n$ is even and $g(n) \in \{1, 2\}$ then $g(2n)$ is the other of $\{1, 2\}$. Thus $g(2n) = 3 - g(n)$ and $g(4n) = g(n)$ for any even $n$. If $n$ is odd then $g(2n) = \text{mex}\{g(2n-1), g(n)\} = \text{mex}\{0, 0\} = 1$. Thus the values along the sequence $2 \cdot 5^{100}$ are 0, 1, 2, 1, 2, . . . , and for $k = 100$ we get $g(10^{100}) = 2$.

Problem 5. There are two piles of chips. A valid move is to take any number of chips from one of the piles (at least one, as in NIM) or to take a single chip from both piles. Find a formula for the Sprague-Grundy value $g(n, 1)$.

Solution. If one of the piles is empty, this is just NIM, so $g(n, 0) = n \oplus 0 = n$. From $(n, 1)$ the possible moves are to $(m, 1)$ with $m < n$, or to $(n, 0)$ or to $(n-1, 0)$. We can find by hand that the first few values of $g(n, 1)$ are 1, 2, 0, 4, 5, 3, 7, 8, 6, . . . Split these to triplets to see the pattern. The general pattern is

$$g(n, 1) = \begin{cases} n + 1 & n \equiv 0 \text{ or } n \equiv 1 \pmod{3}, \\ n - 2 & n \equiv 2 \pmod{3}. \end{cases}$$

To prove this, we check the 3 cases. For example, if $n = 3k$ than all $i < 3k$ are values of $g(m, 1)$ for $m < n$, and $g(3k, 0) = 3k$, so the mex is $3k + 1$. The two other cases are similar.

Bonus Problem 6. For the game from Problem 5:
(a) find a general formula for $g(x, y)$.
(b) Prove your the formula.
Solution. The formula: If \( x, y \in \{0, 1, 2\} \) then \( g(x, y) = (x+y) \mod 3 \). For larger values it turns out that for any \( x, y \leq 3 \) and any \( a, b \in \mathbb{N} \):

\[
g(3a + x, 3b + y) = g(x, y) + 3(a \oplus b).
\]

It is possible to prove this by checking many cases. A shorter proof uses a mixed base 2-3 representation. Write \( x = a_0 + 3 \sum a_i 2^i \), and \( y = b_0 + 3 \sum b_i 2^i \), with \( a_0, b_0 \in \{0, 1, 2\} \) and the others in \( \{0, 1\} \). Then the claim is that

\[
g(x, y) = (a_0 + b_0 \mod 3) + \sum (a_i \oplus b_i) 2^i.
\]

Thus digits of \( x, y \) are added without carry. To prove this is the Sprague-Grundy value, we need to show two things:

- If \( (x', y') \) is a follower of \( (x, y) \) then \( g(x', y') \neq g(x, y) \).
- For every \( s < g(x, y) \) there is a follower with \( g(x', y') = s \).

The first is obvious: changing some digits changes the sum. The second is similar to the case of NIM. If \( s \) differs from \( g(x, y) \) in some digit except the least, then the in first place this occurs \( s \) has digit 0 and \( g(x, y) \) has a 1 (since \( s < g(x, y) \)). This means we can remove chips from one pile to get to value \( s \).

The remaining case is when all but the last digit of \( s \) agree with \( g(x, y) \). Let the last digits of \( s \) and \( g(x, y) \) be \( s_0, c_0 \).

- If \( s_0 = 0, c_0 = 1 \) then one of \( a_0, b_0 \) is non-zero and we take one chip from that pile.
- If \( s_0 = 1, c_0 = 2 \) then also one of \( a_0, b_0 \) is non-zero and we take one chip from that pile.
- If \( s_0 = 0, c_0 = 2 \) and one of \( a_0, b_0 \) is 2, we take two chips from that pile.
- The remaining case is \( s_0 = 0, c_0 = 2, a_0 = b_0 = 1 \). Here we take one chip from each pile.

**Bonus Problem 7.** Write a python program to compute the Sprague-Grundy of a chomp board. After executing your file in python, there should be a function \( \text{chomp}(A) \) which takes a tuple \( A \) and returns the value of that board. For example, the \( 5 \times 5 \) board is \( A=(5,5,5,5,5) \) and \( \text{chomp}(A) \) should return 6. If 3 squares are missing from the top row, this is described as \( (5,5,5,5,2) \). The lengths of rows are given from the bottom up.

Solutions to this problem must be submitted on canvas. I only accept python code, since I run a script to check correctness. A \( 5 \times 5 \) board should not take more than a second.

**Solution.** My solution:

```python
def moves(B): # possible moves from position B, given as a tuple.
    L = len(B)
    for i in range(L):
        for j in range(B[i]):
            if i==j==0: continue
            yield tuple((B[k] if k<i else min(B[k],j)) for k in range(L))

def g(x):
    return mex(g(y) for y in moves(x))
```

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