Problem 1. Recall Grundy’s game, where a move is to split a pile into two unequal piles. If the starting position has piles of sizes 5, 8, 13, find all winning moves.

Solution. We find $g(n)$ for $n \leq 13$ is given by $(0, 0, 1, 0, 2, 1, 0, 2, 1, 0, 2, 1, 3)$. Since $g$ of $(5, 8, 13) = 2 \oplus 2 \oplus 3 = 3$, the winning moves are either to replace the piles of size 5 or 8 by something with NIM-sum 1, or the 13 by something with NIM-sum 0. The corresponding moves are $5 \rightarrow (2, 3)$, $8 \rightarrow (2, 6)$, $13 \rightarrow (5, 8)$.

```python
@memoize
def g(n):
    if n<=2: return 0
    return mex([g(i)^g(n-i) for i in range(1,(1+n)//2)])
```

Problem 2. Consider a partizan subtraction game, where Alice can remove 1, 2, 3, or 4 chips, but Bob can remove at most 3. (So they do not have the same set of available moves.) Determine (with proof) who wins if the game starts with $n$ chips and Alice moves first, and who wins if Bob moves first. (The answer depends on $n$.)

Solution. If $n = 0$ second player wins. If $n \in \{1, 2, 3\}$ first player wins (by taking all the chips).

For larger $n$ Alice wins. To see this, note that if $4|n$ and Bob is first, or if 4 does not divide $n$ and Alice is first, then Alice can win by just playing subtraction with $\{1, 2, 3\}$ chips, never using the extra move she has.

If $n$ is a multiple of 4 and Alice moves first, she takes 4 chips, and is back in the previous case. The same holds if Bob moves first and leaves a multiple of 4.

Problem 3. Consider a game where there are red, blue and green chips. A legal move is to take at most 3 red chips, or at most 4 blue chips, or at most 5 green chips. (Only one colour can be used in a move.) Find the Grundy value of the position $(R, B, G) = (99, 99, 99)$. Find all winning moves from this position.

Solution. The Grundy value for a stack of $n$ red chips is $n \pmod{4}$. The Grundy value for a stack of $n$ blue chips is $n \pmod{5}$, and the Grundy value for a stack of $n$ green chips is $n \pmod{6}$. Therefore the Grundy value for position $(r, b, g)$ is

$$g(r, b, g) = (r \pmod{4}) \oplus (b \pmod{5}) \oplus (b \pmod{6}).$$

For the given position this comes to $3 \oplus 4 \oplus 3 = 4$. A winning move either replace the blue stack with one of value 0, or one of the others with one of value 7. The former is done by taking 4 blue chips. The latter is impossible since the red and green stacks have value at most 5.

Problem 4. The game of friendly frogs is played as follows. There are some marked points in the plane. Two frogs are on two of these points, but can not be on the same point. A player can move one of the frogs in such a way that the distance between them decreases. If there are no valid moves, the player loses. For example, if the frogs are at $(0, 0)$ and $(4, 2)$ it is allowed to move the second to $(3, 3)$ since it is closer to $(0, 0)$. (Since $3^2 + 3^2 < 4^2 + 2^2$.)

Suppose there are 5 points at locations $A = (0, 0)$, $B = (0, 3)$, $C = (2, 1)$, $D = (3, 2)$, $E = (5, 2)$ Find the Grundy value of each position. (There are 10 positions.)
Solution. We first note all the distances:

\[
\begin{align*}
AB &= \sqrt{9} & AC &= \sqrt{5} & AD &= \sqrt{13} & AE &= \sqrt{29} & BC &= \sqrt{8} \\
BD &= \sqrt{10} & BE &= \sqrt{26} & CD &= \sqrt{2} & CE &= \sqrt{10} & DE &= \sqrt{4}
\end{align*}
\]

so the order is \( CD < DE < AC < BC < AB < BD = CE < AD < BE < AE \).

We now calculate the grundy values in this order, noting the followers of each position:

<table>
<thead>
<tr>
<th>position</th>
<th>followers</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>DE</td>
<td>CD</td>
<td>1</td>
</tr>
<tr>
<td>AC</td>
<td>CD</td>
<td>1</td>
</tr>
<tr>
<td>BC</td>
<td>AC, CD</td>
<td>2</td>
</tr>
<tr>
<td>AB</td>
<td>AC, BC</td>
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</tr>
<tr>
<td>BD</td>
<td>CD, DE, BC, AB</td>
<td>3</td>
</tr>
<tr>
<td>CE</td>
<td>CD, DE, AC, BC</td>
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</tr>
<tr>
<td>AD</td>
<td>CD, DE, AC, AB, BD</td>
<td>2</td>
</tr>
<tr>
<td>BE</td>
<td>DE, BC, AB, BD, CE</td>
<td>4</td>
</tr>
<tr>
<td>AE</td>
<td>DE, AC, AB, CE, AD, BE</td>
<td>5</td>
</tr>
</tbody>
</table>

**Bonus Problem 5.** In the previous game, suppose the game starts with player 1 placing one frog, then player 2 places another, then they start moving. (A player can move either frog, not just the one they placed). Where should Player 1 place the first frog?

Solution. From the previous solution we see that \( \{A, B\} \) and \( \{C, D\} \) are P-positions. If Player 1 places a frog in one of A,B then player 2 places a frog in the other and wins. The same holds if player 1 plays in C or D. However, if Player 1 places a frog in E, then player 2 can place the frog in any other vertex and player 1 will win by moving to one of \( \{A, B\} \) or \( \{C, D\} \).