Problem 1. Analyse 3-card Kuhn poker (problem 6.4 from Peres-Karlin).

- There are 3 cards: Jack, Queen and King.
- Each player pays $1 into the pot.
- Each player is dealt one of the cards, and only sees their own card.
- Player I can either pass (P) or bet (B) $1.
  - If player I bets, then player II can either fold (F) or call (C) (adding $1 to the pot).
  - If player I passes, then player II can pass (P) or bet $1 (B). If player II bets, then player I can either fold or call (match the bet).
- If one of the players folds, the other player takes the pot. If neither folds, the player with the high card wins any amount in the pot.

(a) Find an optimal strategy (Nash equilibrium) in this game via reduction to the normal (matrix) form.
(b) Does the resulting strategy involve bluffing (bidding or raising with a low card)?
(c) Does it involve slow playing (passing with a high card)?

Hint: Start by making a list of all possible pure strategies for each player. For example, player 1 could bet with the Q, and then fold if called, and pass with the J or K and if player 2 bets call with the K, fold with the J. For Player 2, the strategy says what to do in each of 6 cases, depending on the card they have and whether Player 1 passed or bet. Then eliminate some dominated strategies before calculating the average outcome for the others. (This still gives a matrix larger than you are used to.) You can use online 0-sum game solvers to find optimal strategies.

Solution. For each card player 1 has they can either bet, or pass. If they pass and player 2 bets, player 1 can call or fold. Denote these three options by B (bet), C (pass+call), and F (pass+fold). Thus a pure strategy is a triplet of such choices for J, Q, K. For example, we can use FBB for the strategy where with the J we pass then fold, and with the Q or K we bet. This gives $3^3$ strategies for player 1. However, with the J is is always better to fold than to call, and with the K better to call than fold, so after removing dominated strategies we have 12 pure strategies:

$$BBB, BBC, BCB, BCC, BFB, BFC, FBB, FBC, FCB, FCC, FFB, FFC.$$ 

For player 2, they can either call (C) or fold (F) after a bet, and either bet (B) or pass (P) after a pass. This gives $4^3 = 64$ strategies, but the same domination (fold with the Jack, call with the King) leaves 16 pure strategies. With the K, Player 2 should bet rather than pass, since if Player 1 calls they win more, and if Player 1 folds it is the same. This leaves 8 strategies for Player 2. For example PFPCBC means to pass/fold with the J, pass/call with Q and bet/call with the K.

With a given pair of strategies, we can find the outcome with each dealt cards, and the expectation. The strategies FBB and PFPCBC give:

- (J,Q): P1 passes, P2 passes, result is -1 for P1.
- (J,K): pass, bet, fold, result is -1
- (Q,J): bet, fold, +1
- (Q,K): bet, call, -2
- (K,J): bet, fold, +1
- (K,Q): bet, call, +2.

Average for this pair of strategies is 0. 

Computing the $12 \times 8$ grid (faster with a computer, but doable by hand), we find some less obvious dominations: For player 1 it is always better to pass+call with the Q then to bet. This is since if P2 has the K they will call a bet, and with the J will fold, but if P1 passes there is a chance P2 will not bet with
the K, or bet with the J. For Player 2, holding the Q it is better to pass than bet (if Player 1 passes). This is since if Player 1 holds the J they will fold and with the K they will call, so for every deal the outcome is better by passing.

This leaves the $8 \times 4$ matrix

$$
\begin{pmatrix}
0 & -2 & 1 & -1 \\
0 & -3 & 2 & -1 \\
1 & -1 & -1 & -3 \\
1 & -2 & 0 & -3 \\
-1 & 0 & 0 & 1 \\
-1 & -1 & 1 & 1 \\
0 & 1 & -2 & -1 \\
0 & 0 & -1 & -1
\end{pmatrix}
$$

This has value -1/3. There are many optimal strategy. One option: Player 1 picks row 3 with probability 1/3 and 5 with probability 2/3. Player 2 picks columns 1,2,3 with probability 1/3 each.

This has Player 1 betting with the J with probability 1/3 (a bluff). However, there are also some equally good strategies that do not involve bluffing, but they do involve slow-play (passing with the K).

**Problem 2.** Consider the following game (which will be discussed on Tuesday):

- A coin is tossed, and Alice sees the result, but Bob does not.
- There are $T$ rounds (with the same coin not changing).
- In each round, each player writes H or T on a piece of paper, then their choices are revealed (at once), but the coin remains hidden.
- At the end of $T$ rounds, Bob pays Alice 1 for each round in which they both picked the same as the hidden coin.

(a) For $T = 1$, write this as a 0-sum game with strategies for each player, and show that the value is 1/2.

(b) For $T = 2$, show that the following strategies give a value of 3/4: Alice picks randomly in round 1, then correctly in round 2. Bob picks randomly in both rounds.

(c) For $T = 3$, show that Alice can guarantee getting at least 1 on average.

(d) Still for $T = 3$, find a strategy for Bob where he pays at most 1 on average.

**Solution.**

(a) For $T = 1$ Alice has 4 options: she can picks always H, always T, always what is on the coin, and always the opposite of the coin. Bob can pick H or T. The matrix is then

$$
\begin{pmatrix}
1/2 & 0 \\
0 & 1/2 \\
1/2 & 1/2 \\
0 & 0
\end{pmatrix}
$$

The third row (always pick what the coin shows) is obviously dominating, and Alice gets 1/2. (It does not matter what Bob does).

(b) In round 1 Alice picks randomly, so if no matter whether Bob picks H or T, she gets on average 1/4 (whenever her pick and the coin both agree with Bob’s pick). In round 2, whatever Bob picks Alice gets 1/2 if the coin agrees with Bob’s pick, and 0 if not. Her total is 3/4 for every pure strategy of Bob, so it is 3/4.

(c) Let Alice ignore the information and play randomly in rounds 1 and 2, and pick the hidden coin in round 3. She receives an average payoff of 1/4 in the first two rounds and 1/2 in the last (no matter what Bob picks), for a total of 1.

(d) Let Bob play randomly in rounds 1 and 2. In round 3, if Alice’s first two moves are HH, Bob picks T. If Alice’s first two moves are TT, Bob picks H. Otherwise Bob plays randomly. Suppose the hidden coin is H. If Alice picks HHH, Bob pays 1/2+1/2+0. If Alice picks HTH or THT, Bob pays 1/2+0+1/2. If Alice picks TTH, Bob pays $0+0+1$. The total is 1 in every case.
Problem 3 (bonus). What happens for higher $T$ in the previous problem? Let $V_T$ be the value. A full solution is to calculate the value with proof. Partial credit for non-matching lower and upper bounds on the value. The following are some directions to consider.

(a) What is $V_3$? Compute the value if you can. Give upper and lower bounds if not.
(b) Prove that $V_T \geq V_{T-1} + \frac{1}{4}$, and therefore $V_T \geq \frac{T+1}{4}$.
(c) Prove that for $T$ even, $V_T \leq \frac{3T}{8}$.
(d) Prove that $\limsup \frac{1}{V_T} \leq \frac{1}{3}$.
(e) Prove that $\limsup \frac{1}{V_T} = \frac{1}{3}$.

Solution.

(a) It turns out that $V_4 = 9/7$.

This is a 0-sum game, so the value can be computed by the minimax theorem. Bob’s possible pure strategies are to pick H or T after each sequence observed from Alice. For Bob’s $k$th move there are $2^{k-1}$ possible observed moves from Alice, and Bob needs to pick H or T after each of these, which gives $2^{1+2+\ldots+2^{T-1}} = 2^{2^{T}-1}$ pure strategies for Bob. For $T = 4$ this are $2^{15}$ pure strategies. For Alice, there is also the hidden coin to consider, so there are $2^{2^{T+1}-2}$ pure strategies.

While the space of strategies is large, it can be reduced significantly. A key idea is that Bob’s actions do not give Alice any new information, so the value does not change if Alice ignores Bob’s choices. (More precisely, for any strategy that Bob picks, any pure strategy of Alice is equivalent to a mixture of strategies that ignore Bob’s actions. Therefore we can (and do) assume that for each value of the hidden coin, Alice has $2^T$ pure strategies, giving the sequence of choices she makes. We can also use symmetry to assume that the distribution Alice uses if the coin is H is the opposite of the distribution if it is T. (So if when the coin is H she picks “HHHT” with probability $p$, then when the coin is T she picks “TTHT” with the same probability.)

Next, Bob can restrict to behavioural strategies (see Chapter 3). This means after each observed sequence from Alice there is a probability of picking H. Thus instead of $2^{15}$ probabilities for $T = 4$ there are only 15 probabilities for a mixed strategy for Bob. By symmetry, we can assume that if bob picks H with probability $x$, after seeing a sequence $a$, then $x_a = 1 - x_a$, where $a^c$ is the complementary sequence. This also means in the first step Bob plays H with probability 1/2, leaving only 7 parameters.

Given such a strategy, we check which of Alice’s pure strategies give the maximal value, and minimize this over Bob’s strategies. We can now use convex optimization on 7 variables to find the value of the game.

The function is

$$f(x) = \max(x_{00}, x_0 + x_{01}, x_0 + x_{10}, x_0 + x_1 + x_{11},$$

$$3/2 - x_{11}, 5/2 - x_1 - x_{10}, 5/2 - x_0 - x_{01}, 7/2 - x_0 - x_0 - x_{00}),$$

and the minimum is attained at

$$x_0 = \frac{4}{7}, \ x_0 = \frac{9}{14}, \ x_1 = \frac{1}{2}, \ x_{00} = 1, \ x_{01} = \frac{9}{14}, \ x_{10} = \frac{5}{7}, \ x_{11} = \frac{3}{14}.$$

Alice receives $9/7$ for any strategy, unless she picks the opposite of the coin in the first 3 rounds.

(b) If Alice ignores the hidden coin in round 1 and uses the $T$ round optimal strategy afterwards, she gets $\frac{1}{4} + V_T$, so this is a lower bound for $V_{T+1}$.

(c) This and the next are a consequence of sub-additivity: $V_{n+m} \leq V_n + V_m$, for any $m, n$. This is the case since Bob can limit Alice to $V_n$ in the first $n$ rounds, then forget what happened in those rounds and limit Alice to $V_m$ in the remaining rounds. Since $V_2 = \frac{3}{4}$, it follows that $V_{2n} \leq \frac{3}{4} n$.

(d) Since $V_3 = 1$ we get $V_{3n} \leq n$. Since $V_n$ is increasing in $n$, the claim follows.

(e) Since $V_T \geq \frac{T}{4}$, the limit is at least 1/4. To get an upper bound, we need to argue that either Alice picks nearly uniform actions for almost all the time, or else she gives up too much information to Bob, and gets nothing later. I do not have a simple way to complete this proof.