1. **10 marks** Consider a game played on the graph below. A token is on a vertex and can move along an arrow.

(a) Calculate the Grundy value for each position.
(b) What are the P-positions?

![Graph](image)

**Solution:**
(a) Calculated from the bottom up, these are: 0,1,0,1; 1,3,2,3,0; 0,1,0,1.
(b) These are a, c, i, j, l.

2. **10 marks** Find the value and optimal strategies for the following game, for every $t \in \mathbb{R}$:

$$A = \begin{pmatrix} t & 0 \\ 4 & 2t \end{pmatrix}$$

**Solution:**
- If $t \in [0, 2]$, then $A_{2,2}$ is a saddlepoint and the value is $2t$.
- If $t \in [2, 4]$ then $A_{2,1}$ is a saddlepoint and the value is 4.
- Otherwise, equalizing payoffs gives $tx_1 + 4x_2 = 2tx_2 = v$. Thus $x = (\frac{2t-4}{3t-4}, \frac{t}{3t-4})$ and $v = \frac{2t^2}{3t-4}$. Similarly, equal entries of $A_y$ gives $y = (\frac{2t}{3t-4}, t - 43t - 4)$.

3. **10 marks** Find the value and an optimal strategy for each player in the following games:

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 1 & 2 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 6 & 4 & 3 \\ 4 & 4 & 0 & 1 \\ 2 & 2 & 4 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}$$
Solution: For $A$ we find that the worst outcome from strategy $x = (x_1, 1-x_1)^T$ is $\min(1 + 3x_1, 2, 6 - 6x_1)$. This is in column 1 if $x_1 \leq 1/3$, and column 3 if $x_1 \geq 2/3$, and in the center if $x_1 \in [1/3, 2/3]$. The maximum of this is 2 for any $x_1 \in [1/3, 2/3]$, so the value is 2 and any such $x$ is optimal for player 1. Player 2 must use column 2.

For $B$, column 2 is dominated by 1, row 4 by row 3. After removing these, row 1 is dominated by row 3, and then column 4 by column 3. This leaves $A' = \begin{pmatrix} 4 & 0 \\ 2 & 4 \end{pmatrix}$. Equalizing payoffs we find the optimal strategies are $x = y = (1/3, 2/3)$ and the value is $8/3$.

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4. **10 marks** Consider the following 0-sum game. Each player picks a number from \{0, 1, 2, 3\}. Call the selections $i, j$. If $i+j$ is even player 1 gets $ij$. Otherwise player 2 gets $ij$ (so player 1 pays $ij$).

(a) Write the game in matrix form.
(b) Find the value of the game, and an optimal strategy for each player.
(c) Find a different optimal strategy for player 1.

**Solution:**

(a) The matrix form is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & -2 & 4 & -6 \\ 0 & 3 & -6 & 9 \end{pmatrix}.$$  

(b) $A_{1,1}$ is a saddlepoint, and the value is 0. Both players can guarantee that by picking 0.

(c) Player 1 can get on average 0 by picking any strategy $x$ with $2x_2 = x_1 + 3x_3$. For example $(0, 2/3, 1/3, 0)^T$. This is since the rows of $A$ are all proportional to each other.

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5. **10 marks** Consider the subtraction game with set $A = \{1, 3, 6\}$.

(a) Find the Grundy value for piles of size $n \leq 16$.
(b) If we play the sum of 4 such games with 4 piles of sizes 5, 6, 7, 8, list all the winning moves for player 1.

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Solution:

(a) The first few values are:

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| 0 | 1 | 0 | 1 | 0 | 1 | 2 | 3 | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 2 | 3 |

The pattern is 010101232 repeated.

(b) The values of the piles are 1, 2, 3, 2 with $1 \oplus 2 \oplus 3 \oplus 2 = 2$. The winning moves are to replace a pile with value 1 by a pile with value 3 (impossible), or a pile of value 2 by one of value 0 (moves 6 $\rightarrow$ 0, 8 $\rightarrow$ 2) or the pile of value 3 to value 1 (move 7 $\rightarrow$ 1).