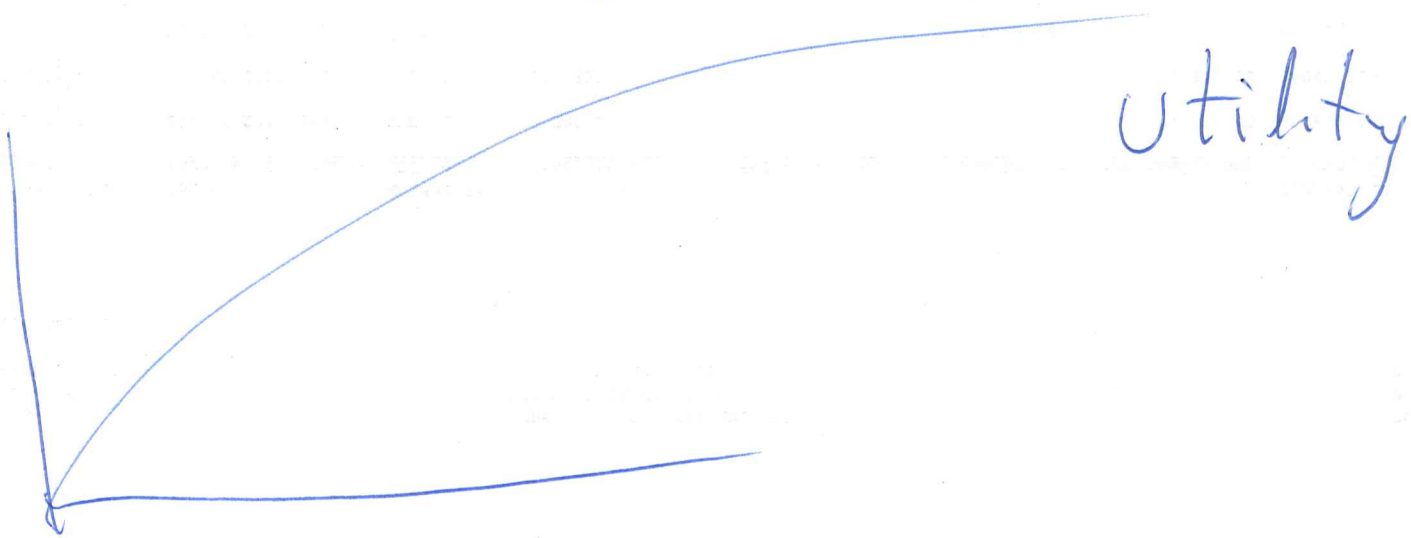
20210908

ABC :	100	35	35	30
AB :	80	45	45	10
AC :	75	—	35	35
BC :	70			

[Allocation games]



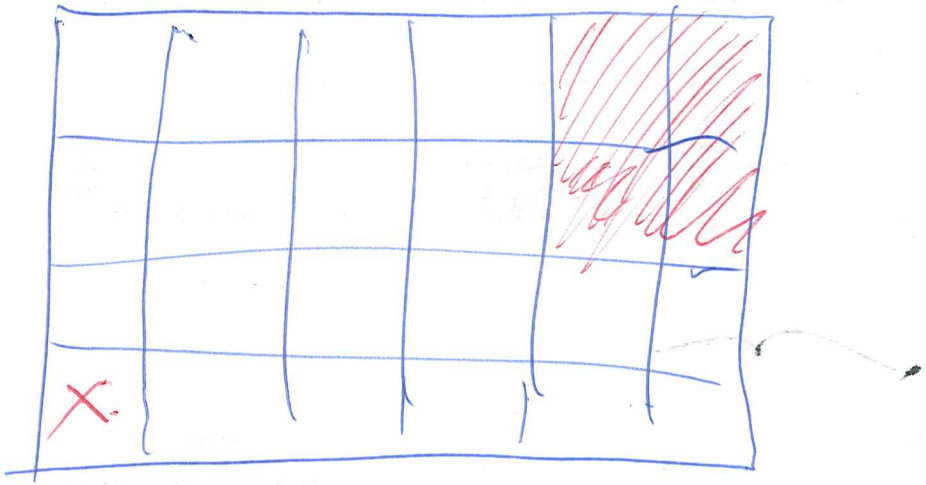
Combinatorial games

- Outcome : win / lose.
- 2 players
- Full information
- No randomness

e.g. CHOMP.

remove all R/Ab. a sq.

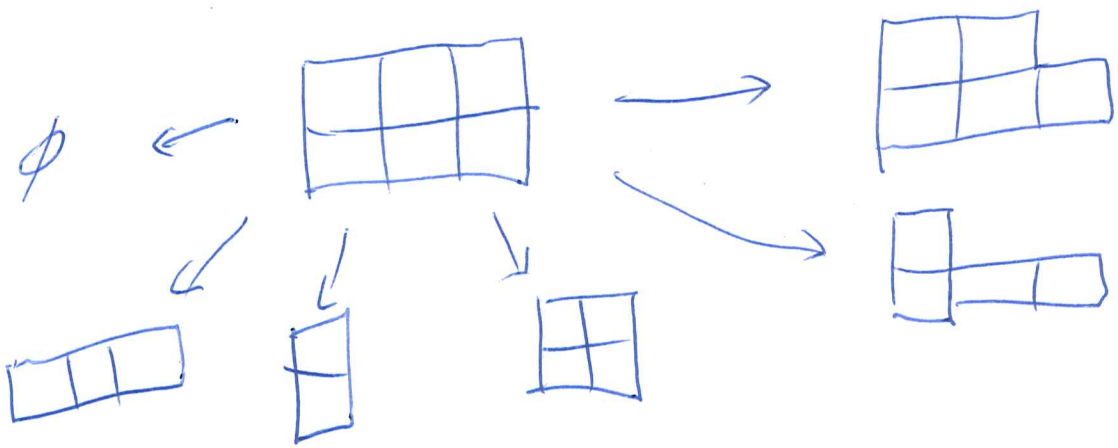
lose if forced to pick X.



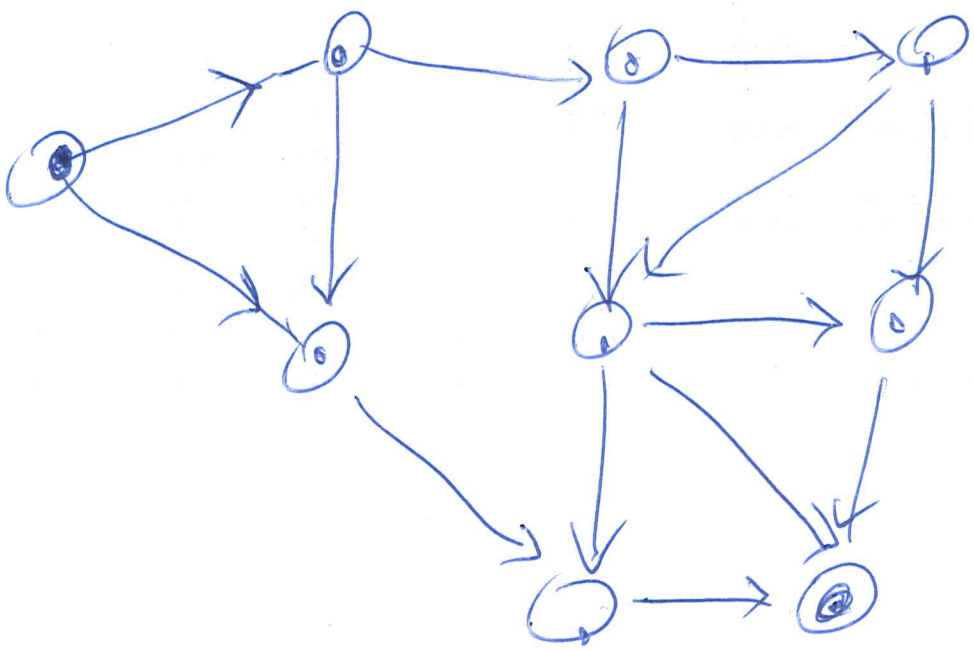
def: A game is <u>progressively bounded</u> if for any state x of the game, there is a bound $M(x)$ on the number of moves before game ends. (no matter how players move)

Def: <u>Game</u>: ⊛ Set of states / positions

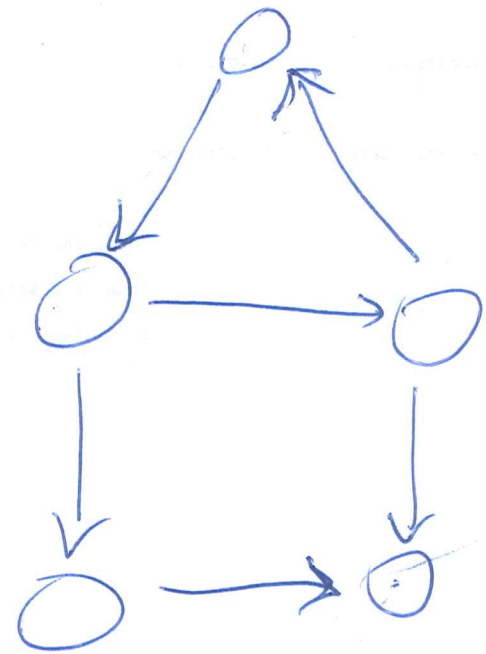
⊛ Valid moves from each state.



* Outcome: At some states game ends.
One player is winner.



prog. bdd.



not prog. bdd.

<u>Theorem</u>: In any prog. bdd game, with no draw at any pos. x, either player 1 or player 2 has a strategy that guarantees they win.

[players alternate moves, combinatorial game]

20210910

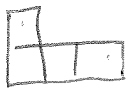
Combinatorial games

S : set of game states.

game state = everything that can effect the game.

F : followers: for each state $x \in S$

$F(x)$ = set of states allowed after x.

e.g. in CHOMP $F($  $) = \{$  ,  ,  , $\emptyset \}$

<u>Impartial</u>: both players have same moves.

<u>partizan</u>: Each player has different set of moves.

F_1 , F_2 for player 1, 2.

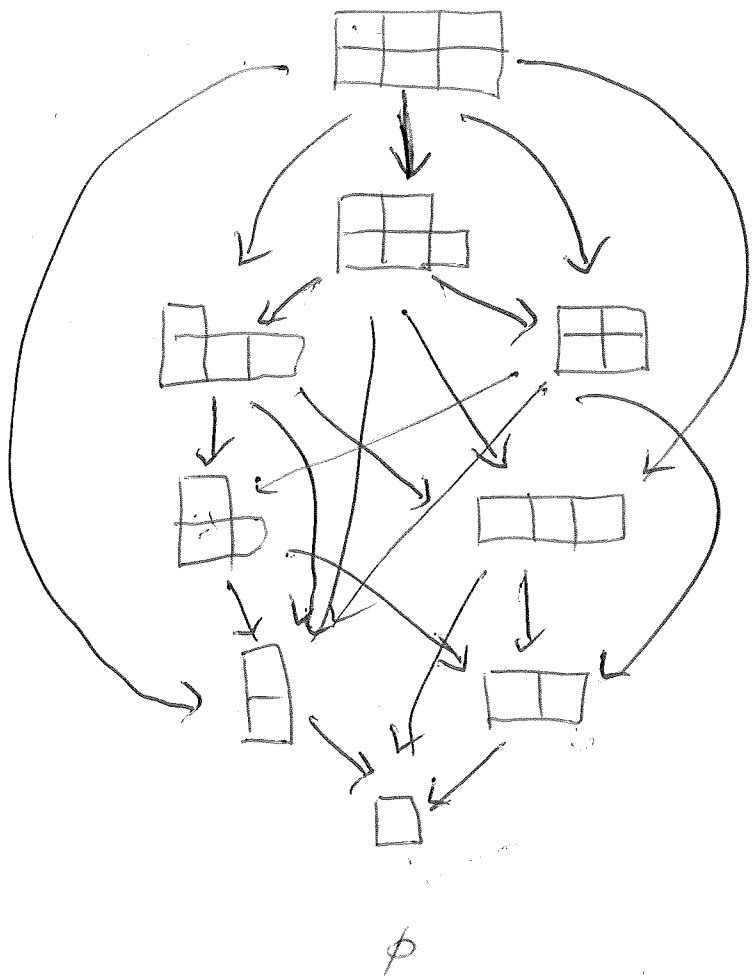
Ending: Some states specify a winner / loser

<u>Normal play</u>: If you cannot move, you lose.

<u>Misére play</u>: If you have no move, you win.

<u>Game graph</u>: $x \to y$ if $y \in F(x)$

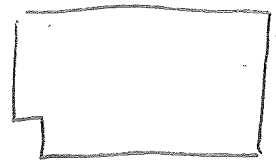
e.g 2×3 CHOMP:



can always reach $\emptyset$

Reduce Misere to normal play:

If a move ends the game and the mover loses, forbid this move.

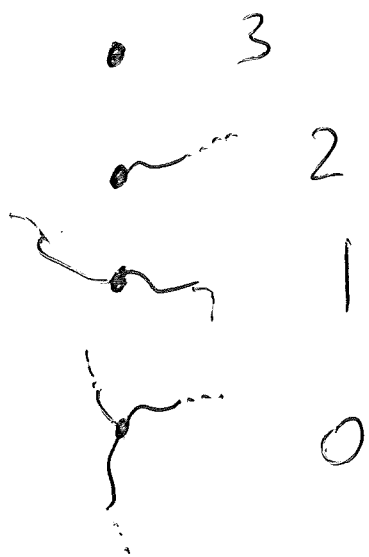
In CHOMP: Play on 

<u>Recall</u>: a game is Progressively bounded (P.B.) if $\forall x \ \exists M(x)$ s.t. the game started at x lasts at most $M(x)$ moves.

e.g. <u>ARCS</u>

Claim: ARCS is P.B.

Proof: Consider total # of open connections.



A move reduces 1 conn. at 2 pts adds one pt with one conn. left

Total decreases by 1 in each move. □

eg start with n pts then $M \leq 3n-1$

exercize: $3n-1$ is possible.

<u>Def</u>: A strategy is a rule to pick a move at every state of the game.

Given strategies for P1, P2 can determine the outcome by following the moves prescribed.

<u>Def</u>: A winning strategy is one that wins no matter what opponent does.

<u>Claim</u>: Impossible for both players to have winning strategies.

<u>Proof</u>: Play the 2 winning strat. against each other. One must lose.

<u>Theorem</u>: In a prog. bdd game, one of the players must have a winning strategy.

[Assuming outcome is only win/lose, combinatorial game]

Can we have a game where each player has 2 strat.

P1: A,B

P2: C,D

$$\begin{array}{c|cc} & C & D \\ \hline A & 1 & 2 \\ B & 2 & 1 \end{array}$$

← who wins given choice of strat.

(matrix form of a game)

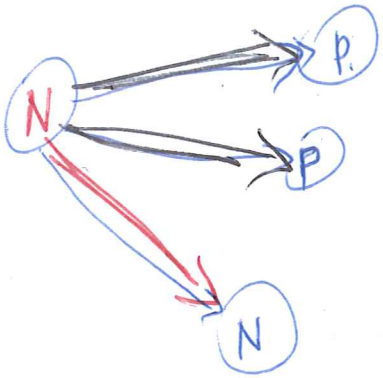
$\boxed{\text{This is not a comb. game!}}$

In this game, neither has a winning strat.

<u>Thm</u> In any combinatorial game, prog. bdd, for any x one of the players has a winning strat.

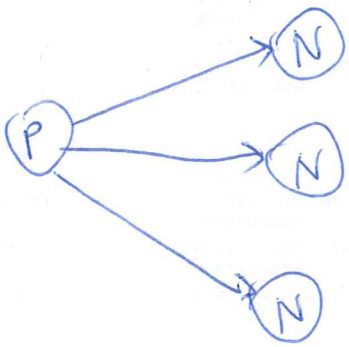
<u>Def</u>: x is a N-pos. if starting at x P.1 has a winning strat. (next player)

P-pos. if " " " P.2 can win. (prev. player)



$\longrightarrow$ = good move.

$\longrightarrow$ bad move.



- x is a P-pos. if and only if every $y \in F(x)$ is an N-pos.
- x is an N-pos. if and only if some (exists) ≥ 1 $y \in F(x)$ is a P-pos.

Let N_i = positions where next player can win in $\leq i$ moves.

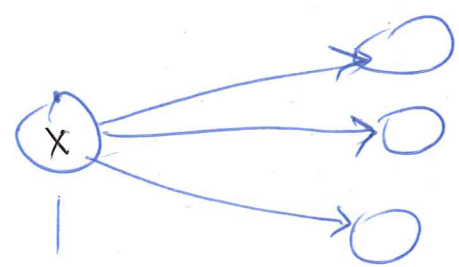
P_i = same for 2nd player.

<u>Claim:</u> If from x the game lasts $\leq i$ moves then $x \in P_i$ or $x \in N_i$

<u>Pf</u>: Induction on i.

<u>$i=0$</u> : trivial.

If true for all $i \leq n$, lets prove for $i=n$



from here $\leq n$ moves

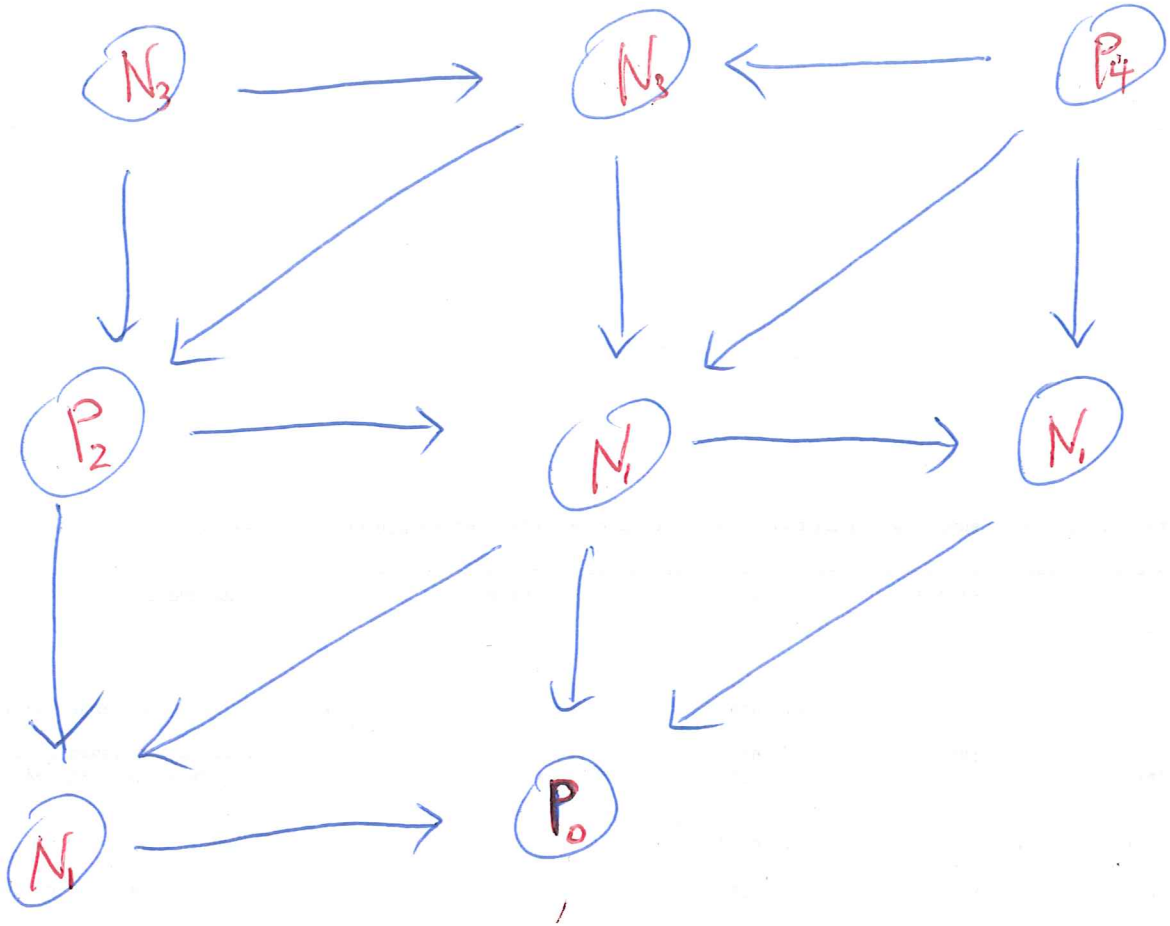
from here $\leq n-1$ moves

By induction hyp. all $y \in F(x)$ are in $P_{n-1} \cup N_{n-1}$

If all are N_{n-1} then $x \in P_n$

If not, some $y \in F(x)$ is $y \in P_{n-1}$

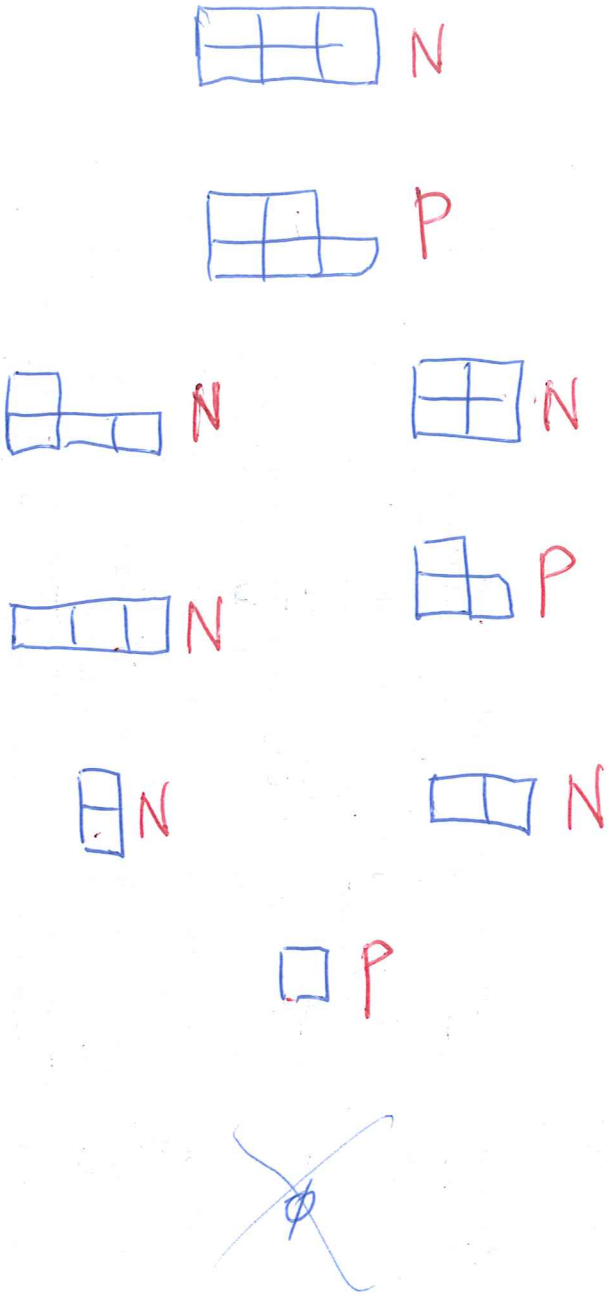
First player wins in $\leq n$ moves, starting by moving to y. $\square$



Normal play.

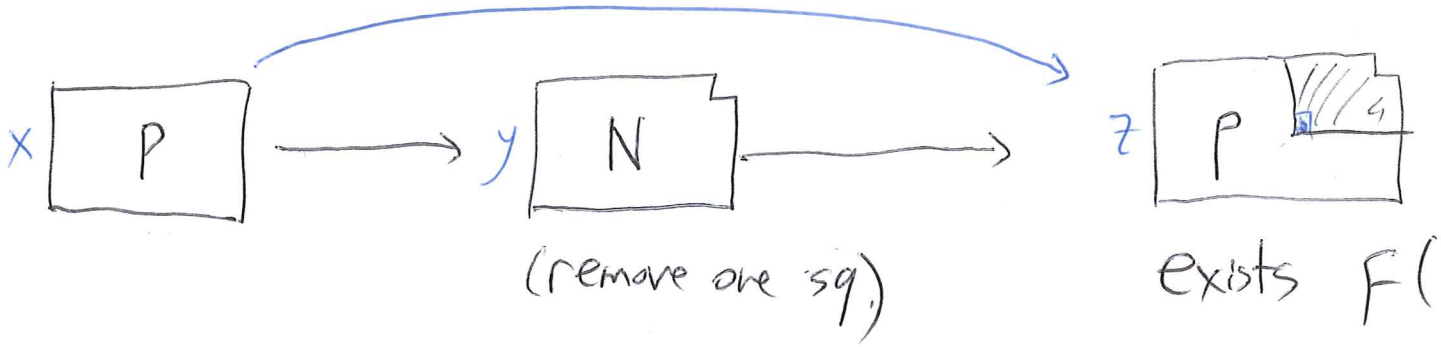
$P_0 \subset P_1 \subset P_2 \cdots$

2x3 CHOMP



Thm: $n \times m$ chomp board is an N-pos $\forall n,m \neq (1,1)$

Proof: Assume not, and get contradiction.



(remove one sq.)

exists $F(\square) \in P$

contradiction since $z \in F(x)$, so z must be N-pos.

$y = x$ minus single square

20210915

SUBTRACTION

Legal move: remove between 1 and 4 beads from a pile.

<u>Claim</u>: P-positions are $\{0, 5, 10, 15, \ldots\} = 5\mathbb{N}$

N-pos. are the rest: $\mathbb{N} \setminus (5\mathbb{N})$

To prove this need to show 2 things:

- if $n \in 5\mathbb{N}$ then $F(n)$ are all in $\mathbb{N} \setminus (5\mathbb{N})$
- If $n \notin 5\mathbb{N}$ then $\exists y \in F(n)$ with $y \in 5\mathbb{N}$.

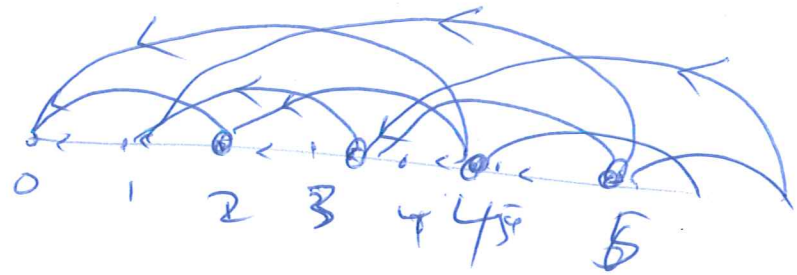
1st: easy.

2nd: If $5 \nmid n$ then $n = 5x + r$ with $r \in \{1, 2, 3, 4\}$

$y = 5x$ is the follower.

Consider subtraction with allowed moves set $S = \{1, 2, 4\}$

	type	foll.
0	P	—
1	N	0
2	N	0, 1
3	P	1, 2
4	N	0, 2, 3
5	N	1 3 4
6	P	2, 4, 5



Guess: $P = 3\mathbb{N}$

<u>Proof:</u> If $n = 3x$ then $F(n) = \{3x-1, 3x-2, 3x-4\}$
none is divisible by 3.

If n not a mult. of 3 then $n = 3x+1$ or $3x+2$
$3x$ is a follower of n.

e.g. $S = \{2, 3, 5\}$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
type	P	P	N	N	N	N	N	P	P	N	N	N	N
			0	0	1	0	0	4,5,2	3,5,6	7	7	8	7

repeats

<u>Claim</u>: N-pos. are $(7\mathbb{N}) \cup (7\mathbb{N}+1)$

<u>Proof</u>: 7 cases.

e.g. If $n = 7x+1$ followers are $7x-1, 7x-2, 7x-4$ are all N-pos.

If $n = 7x+5$ then $7x$ is a p-pos and follower.

<u>Theorem</u>: For any finite set S, the types are eventually periodic.

$S = \{4, 5, 7\}$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
P	P	P	P	N	N	N	N	N	N	N	P	P	P	P	

<u>note</u>: can have initial types that do not fit the pattern.

<u>Proof</u>: Suppose $\max(S) = m$.

Type(n) is determined by types of $(n-m, \ldots, n-1)$

Given the vector of types of $(n+1 \ldots n+m)$ can find the vector for $(n+2 \ldots n+m+1)$

This vector has 2^m possibilities.

So it must repeat.

$$\overset{n}{\underbrace{PNNPNP}}\,x\,u \;\overset{p}{\cdots}\; \overset{t}{\underbrace{PNNPNP}}\,y\,v \;\overset{p}{\cdots}\; \underbrace{PNNPNP}$$

$=$ then $x=y$

$u=v$

$\vdots$

If types of $n+1 \ldots n+m$ same as types of $t+1 \ldots t+m$, then type of $n+m+1$ $=$ type of $t+m+1$

$n+m+2$ $\quad$ $t+m+2$

$\vdots$ $\quad$ $\vdots$

for all $i \geq 0$: $\quad$ $n+m+i$ $\quad$ $t+m+i$

If $p = t-n$ then get period p from n onward $\square$

note: cannot determine type(n) from types(n+1 ... n+m)

For example, $S = \{2,4,7\}$:

n	0	1	2	3	4	5	6	7	8	9	10	11
Type	P	P	N	N	N	N	P	N	N	P	N	N

repeats

$\{2,7,8\}$ $\{3,7,8\}$ $\{1,6,9\}$ are other examples where period starts later.

<u>NIM</u>: can take any number of chips, at least 1 from a single pile

Claim: 1 pile is N pos.

2 piles are P-pos if they are equal,

N-pos. if not.

20210917

Recall NIM:

Several piles of chips.

Valid move: take any number of chips from a single pile.

Normal play: Take last chip to win.

One pile: any $n \neq 0$ is an N-pos.

Two piles: (n,m) is P-pos if and only if $n=m$

<u>Proof</u>: To show that P-positions are A and N-pos. are $B = A^c$ need to show:

$\forall x \in A$ all followers in B and

$\forall x \in B$ has ≥ 1 follower in A.

Base 2: write n as sum of powers of 2:

$$n = 2^a + 2^b + 2^c + \dots$$

$$\begin{array}{ccccccc} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \end{array} = 2^6 + 2^4 + 2^3 + 2^1 + 2^0 = 64 + 16 + 8 + 2 + 1$$

$$= 91$$

NIM-sum (exclusive or)

write in base 2,
add without carries.

$\oplus$

e.g. $6 \oplus 23$:

$6 = 4+2 = 110$ base 2

$23 = 16+4+2+1 = 10111$ base 2

$$\begin{array}{r} 110 \\ 10111 \\ \hline 10001 \end{array}$$

$16+1 = 17$

$6 \oplus 23 = 17$

$21 \oplus 28 = 2^3 + 2^0 = 9$

$21 = 16+4+1$

$28 = 16+8+4$

$$\begin{array}{r} 10101 \\ 11100 \\ \hline 01001 \end{array}$$

$21 \oplus 28 \oplus 5 = 12$

$$\begin{array}{r} 10101 \\ 11100 \\ 101 \\ \hline 01100 \end{array}$$

$2^3 + 2^2 = 12$

Facts: NIM sum is commutative and associative:

$$a \oplus b = b \oplus a$$

$$(a \oplus b) \oplus c = a \oplus (b \oplus c) \quad \text{written as} \quad a \oplus b \oplus c$$

$$1 \oplus 2 \oplus 3 = 0$$

$$\begin{array}{r} 1 \\ 10 \\ 11 \\ \hline 00 \end{array}$$

$$n \oplus n = 0$$

Theorem: $(n_1 \ldots n_k)$ is a P-pos. in NIM if and only if $n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$.

e.g. $(6, 9, 10, 11, 12)$:

NIM-sum is 2

$$\begin{array}{r} 110 \\ 1001 \\ 1010 \\ 1011 \\ 1100 \\ \hline 0010 \end{array}$$

winning moves:

take 2 from 6 or 10 or 11

Proof: ① If $n_1 \oplus \cdots \oplus n_k = 0$ then followers have non-zero NIM-sum.

Changing a single n_i to $x \neq n_i$ changes at least one digit in base 2, so that column no longer has an even sum.

② If $\underbrace{n_1 \oplus \cdots \oplus n_k}_{s} \neq 0$ need a follower with NIM-sum 0.

e.g.

	1	0	1	1	1	0	1	n_1
	1	0	1	1	0	1	0	⋮
	1	0	0	0	1	0	1	⋮
	1	1	0	1	1	1	0	⋮
	1	0	1	0	0	0	1	n_k
	0	1	0	0	1	1	0	s

Note: $n_1 \oplus \cdots \oplus n_k \oplus s = 0$

$$n_1 \oplus \cdots \oplus (n_i \oplus s) \oplus \cdots \oplus n_k = 0$$

so winning moves are $n_i \longrightarrow n_i \oplus s$ for any i

Legal iff $n_i \oplus s < n_i$

If largest 1 in s in position k, then some n_i has 1 in that position.

Can move from those piles.

$$\begin{array}{ll} 10001011 & n_i \\ 10011011 & s \\ \hline 00010000 & n_i \oplus s \end{array}$$

$n_i \oplus s$ has 0 in pos. k

so $n_i \oplus s < n_i$

$\square$