ABC : 100
AB  : 80
AC  : 75
BC  : 70

35  35  30
45  45  10
---  35  35

[Alocation games]

utility
Combinatorial games

- Outcome: win/lose
- 2 players
- Full information
- No randomness

e.g. CHOMP.
remove all R/Ab.
a sq.
lose if forced to pick X.
def: A game is progressively bounded if for any state x of the game, there is a bound M(x) on the number of moves before game ends. (no matter how players move)

Def: Game: Set of states / positions
  - Valid moves from each state.

[Diagram of game states and moves]
Outcome: At some states game ends. One player is winner.
Theorem: In any pos. 2td game, with no draw at any pos. $x$, either player 1 or player 2 has a strategy that guarantees, combinatorial game, players alternate moves.
Combinatorial games

\( S \): set of game states

- game state = everything that can affect the game

\( F \): followees; for each state \( x \in S \)

\[ F(x) = \text{set of states allowed after } x \]

- e.g. in CHOMP \( F(\square) = \{ \text{\square, \square, \square, \emptyset} \} \)

Impartial: both players have same moves

Partisan: each player has different set of moves

\( F_1, F_2 \) for player 1, 2

Ending: some states specify a winner / looser
Normal play: If you cannot move, you lose.

Misère play: If you have no move, you win.

Game graph: $x \rightarrow y \iff y \in F(x)$

E.g., $2 \times 3$ Chomp:

Can always reach $\emptyset$
Reduce Misere to normal play:
If a move ends the game and the mover loses, forbid this move.

In Chomp: Play on

Recall: a game is progressively bounded (P.B.) if
\( \forall x \in M(x) \) s.t. the game started at \( x \), lasts at most \( M(x) \) moves.

e.g. ARCS
Claim: ARCS is P.B.

Proof: Consider total # of open connections.

- 3
- 2
- 1
- 0

A move reduces 1 conn. at 2 pts adds one pt with one conn. left

Total decreases by 1 in each move.

E.g., start with N pts then M ≤ 3N - 1

Exercise: 3N - 1 is possible.
Def: A strategy is a rule to pick a move at every state of the game.

Given strategies for $p_1$, $p_2$ can determine the outcome by following the moves prescribed.

Def: A winning strategy is one that wins no matter what opponent does.

Claim: Impossible for both players to have winning strategies.

Proof: Play the $2$ winning strats against each other, one must lose.
Theorem: In a prog. I&D game, one of the players must have a winning strategy. [Assuming outcome is only win/lose, combinatorial game]
Can we have a game where each player has 2 strat.

P1: AB
P2: CP

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

← who wins given choice of strat.

(matrix form of a game)

In this game, neither has a winning strat.

This is not a comb. game!

Thm: In any combinatorial game, prog. bdd, for any x one of the players has a winning strat.

Def: x is a N-pos. if starting at x P<sub>1</sub> has a winning strat. (next player)

P-pos. if x is P<sub>2</sub> can win. (prev. player)
\[ \rightarrow = \text{good move}. \]
\[ \text{- bad move}. \]

\( x \) is a p-pos. if and only if
\( \forall y \in F(x) \) is an N-pos.

\( x \) is an N-pos. if and only if some (exists) \( y \in F(x) \) is a p-pos.

Let \( N_i \) = positions where next player can win in \( \leq i \) moves.

\( P_i \) = same for 2nd player.
Claim: If from $x$ the game lasts $\leq i$ moves then $x \in P_i$ or $x \in N_i$.

Proof: Induction on $i$.

$i = 0$ : trivial.

If true for all $i \leq n$, let's prove for $i = n$.

By induction hypothesis, all $y \in F(x)$ are in $P_{n-1} \cup N_{n-1}$.

If all are $N_{n-1}$, then $x \in P_n$.

If not, some $y \in F(x)$ is $y \in P_{n-1}$.

First player wins in $\leq n$ moves, starting by moving to $y$. ∎
Normal play.

P₀ c P₁ c P₂ ...
2x3 CHOMP
Thm: \( n \times m \) chomp board is an \( N \)-pos \( \forall n,m \neq (1,1) \)

Proof: Assume not, and get contradiction.

\[ x \xrightarrow{P} y \xrightarrow{N} z \]

(remove one sq.)

\[ \text{exists } f(\square) \in P \]

contradiction since \( z \in F(x) \), so \( z \) must be \( N \)-pos.

\[ y = x \text{ minus single square} \]
SUBTRACTION

Legal move: remove between 1 and 4 beads from a pile.

Claim: P-positions are \( \{0, 5, 10, 15, \ldots \} = 5N \)
N-positions are the rest: \( 5N \setminus (5N) \)

To prove this need to show 2 things:
- if \( n \in 5N \) then \( F(n) \) are all in \( 5N \setminus (5N) \)
- if \( n \not\in 5N \) then \( \exists y \in F(n) \) with \( y \in 5N \)

1st: easy.

2nd: if \( 5 \nmid n \) then \( n = 5x + r \) with \( r \in \{1, 2, 3, 4\} \)
\( y = 5x \) is the follower.
Consider subtraction with allowed move set $S = \{1, 2, 4\}$

<table>
<thead>
<tr>
<th>type</th>
<th>follow</th>
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<tbody>
<tr>
<td>0</td>
<td>P</td>
</tr>
<tr>
<td>1</td>
<td>N</td>
</tr>
<tr>
<td>2</td>
<td>N</td>
</tr>
<tr>
<td>3</td>
<td>P</td>
</tr>
<tr>
<td>4</td>
<td>N</td>
</tr>
<tr>
<td>5</td>
<td>N</td>
</tr>
<tr>
<td>6</td>
<td>P</td>
</tr>
</tbody>
</table>

**Guess:** $P = 3IN$

**Proof:** If $n = 3x$ then $F(n) = \{3x-1, 3x-2, 3x-4\}$

None is divisible by 3.

If $n \neq$ not a multiple of 3 then $n = 3x+1$ or $3x+2$

$3x$ is a follower of $n$. 
e.g. \( S = \{2, 3, 5\} \)

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>P</td>
<td>P</td>
<td>N</td>
<td>N</td>
<td>N</td>
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<td>P</td>
<td>P</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
</tbody>
</table>

repeats.

Claim: \( p \)-pos. are \( (7N) \cup (7N+1) \)

Proof: 7 cases.

- e.g. If \( n = 7x+1 \) followers are \( 7x-1, 7x-2, 7x-4 \) are all \( N \)-pos.

- If \( n = 7x+5 \) then \( 7x \) is a \( p \)-pos and follower.
Theorem: For any finite set $S$, the types are eventually periodic.

$$S = \{4, 5, 7\}$$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
PPPPPNNNNNNNNNNNNPPPPPP

Note: Can have initial types that do not fit the pattern.

Proof: Suppose $\max(S) = M$.

Type $(n)$ is determined by types of $(n-M, \ldots, n-1)$.

Given the vector of types of $(n+1, \ldots, n+m)$ can find the vector for $(n+2, \ldots, n+m+1)$.
This vector has $2^m$ possibilities. So it must repeat.

\[ n \quad p \quad t \quad p \quad \]

\[
\text{PNNPNP}_1 \times 0 \quad \text{PNNPNP}_1 \times 1 \quad \text{PNNPNP}_1
\]

\[ = \text{ then } x = y \]
\[ u = v \]
\[ \vdotss \]

If types of \( n+1 \ldots n+m \) are same as types of \( t+1 \ldots t+m \), then type of \( n+m+1 = \text{type of } t+m+1 \)
\[ n+m+2 \quad t+m+2 \]
\[ \vdots \]

for all \( i > 0 \):
\[ n+m+i \quad t+m+i \]

If \( p = t - n \) then get period \( p \) from \( n \) onward
note: cannot determine type (n) from types (n+1...ntm)

For example, S = {2, 4, 7}:

0 0 1 2 3 4 5 6 7 8 9 10

Type: P P N N N N P N P N N

[2, 7, 8], [3, 7, 8], [1, 6, 9] are other examples where period starts later.
NIM: can take any number of chips, at least 1 from a single pile.

Claim: 1 pile is N pos.
2 piles are p-pos if they are equal, N-pos. if not.
Recall NIM:
Several piles of chips.
Valid move: take any number of chips from a single pile.
Normal play: Take last chip to win.

One pile: any \( n \neq 0 \) is an N-pos.
Two piles: \((n,m)\) is P-pos if and only if \( n=m \)

Proof: To show that P-positions are A and N-pos.
are \( B = A^c \) need to show:
\( \forall x \in A \) all followers in B and
\( \forall x \in B \) has \( \geq 1 \) follower in A.
Base 2: write \( n \) as sum of powers of 2:

\[ n = 2^a + 2^b + 2^c + \ldots \]

\[ 1011011 = 2^6 + 2^4 + 2^3 + 2^1 + 2^0 = 64 + 16 + 8 + 2 + 1 = 91 \]
NIM-sum (exclusive or)

write in base 2,
add without carries.

e.g. $6 \oplus 23 :$

\[
\begin{array}{c}
110 \\
10111 \\
10001 \\
\hline
16 + 1 = 17
\end{array}
\]

$6 \oplus 23 = 17$

\[
\begin{array}{c}
6 = 4 + 2 = 110 \text{ base 2} \\
23 = 16 + 4 + 2 + 1 = 10110 \text{ base 2}
\end{array}
\]

\[
\begin{array}{c}
21 \oplus 28 = 2^3 + 2^2 = 9 \\
21 = 16 + 4 + 1 \\
28 = 16 + 8 + 4
\end{array}
\]

\[
\begin{array}{c}
10101 \\
\hline
01001
\end{array}
\]

\[
21 \oplus 28 \oplus 5 = 12
\]

\[
\begin{array}{c}
10101 \\
11100 \\
\hline
101
\end{array}
\]

\[
\begin{array}{c}
01100 \\
\hline
2^3 + 2^2 = 12
\end{array}
\]
Facts: NIM sum is commutative and associative:

\[ a \oplus b = b \oplus a \]

\[ (a \oplus b) \oplus c = a \oplus (b \oplus c) \] written as \( a \oplus b \oplus c \)

\[ 1 \oplus 2 \oplus 3 = 0 \]

\[ \begin{array}{c}
1 \\
10 \\
11 \\
00 \\
\end{array} \]

\[ n \oplus n = 0 \]

Theorem: \((n_1, \ldots, n_k)\) is a P-pos. in NIM if and only if \(n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0\).

E.g. \((6, 9, 10, 11, 12)\):

\[ \begin{array}{c}
110 \\
1001 \\
1010 \\
1011 \\
1100 \\
0010 \\
\end{array} \]

NIM-sum is 2.

Winning moves: take 2 from 6 or 10 or 11.
Proof: 1) if \( n, \theta, \ldots, \theta n_k = 0 \) then followers have non-zero NIM-sum.

Changing a single \( n_i \) to \( x \neq n_i \) changes at least one digit in base 2, so that column no longer has an even sum.

2) If \( n, \theta, \ldots, \theta n_k = 0 \) need a follower with NIM-sum 0.

e.g. 

\[
\begin{array}{ccccccc}
10111101 & \vdots & n_i \\
10110100 & \vdots & n_i \\
1000101 & \vdots & n_i \\
11011110 & \vdots & n_i \\
101000101 & \vdots & n_k \\
\hline
010011011 & \vdots & S
\end{array}
\]
Note: $\mathbf{n}_1 \oplus \cdots \oplus \mathbf{n}_k \oplus \mathbf{s} = 0$

$\mathbf{n}_1 \oplus \cdots \oplus (\mathbf{n}_i \oplus \mathbf{s}) \oplus \cdots \oplus \mathbf{n}_k = 0$

so winning moves are $\mathbf{n}_i \rightarrow \mathbf{n}_i \oplus \mathbf{s}$ for any $i$.

Legal iff $\mathbf{n}_i \oplus \mathbf{s} < \mathbf{n}_i$.

If largest 1 in $\mathbf{s}$ in position $k$, then some $\mathbf{n}_i$ has 1 in that position.

Can move from those piles.

\[
\begin{array}{cccc}
10001011 & & n_i & \\
10011011 & & s & \\
\hline
00010000 & & \mathbf{n}_i \oplus \mathbf{s} & \\
\end{array}
\]

$n_i \oplus \mathbf{s}$ has 0 in pos $k$ so $\mathbf{n}_i \oplus \mathbf{s} < \mathbf{n}_i$.