Today: sums of games

Idea: play several games at Once

e.g. SUBTRACTION and CHOMP

- one option a move is a move in each of the games
- move in one of the games of your choice
- move in a pre-determined game
Winning condition:

- win all games
- win most games
- first to win a game wins

Disjunctive sum of games move in one of the games

If there is no available move – you lose.
Given several Impartial P.B. games, what are the \( p \)-positions in their sum?

E.g. \( \text{SUBTR. } \{1, 2\} \)

\( p \)-pos. are \( \{3n\} \)

Play 2 copies at once:

\[
\begin{array}{c}
14 \\
18
\end{array}
\]

Idea: move to 12, 18
Games $G_1, G_2, \ldots$

$G_1 + G_2 + \ldots + G_k$ is the disj. sum.

Positions are $(x_1, \ldots, x_k)$ with $x_i$ a pos. in $G_i$.

Claim: If for every $i$

$x_i$ is $P$-pos. in $G_i$

then $(x_1, x_2, \ldots, x_k)$ is a $P$-pos.
Proof: If next player move $x_i \rightarrow y_i$, then $y_i$ is N-pos.

Then $y_i$ has follower $z_i$ which is a P-pos.

You move $y_i \rightarrow z_i$

$(x_i - x_{i-1}, z_i, x_{i+1}, ..., x_k)$

You make the last move in every game.
14, 18 move to 12, 18
15, 18 is p-pos.
(hope opp makes an error)

\((14, 17)\) has type \((N,N)\)
\((P,P) \implies P\)-pos,
\((P,N) \text{ or } (N,P) \implies N\)-pos
\((1, 1)\) is P-pos.
\((1, 2)\) is N-pos.

If \(xy\) are N-pos,
\((x,y)\) can be either N or P.
NIM with k piles is the sum of k games with a single pile. Each game is 1-pile NIM (SUBTRACTION \{1,2,3,\ldots\})

Sprague-Grundy theory:

Let G be a game, P.B. Impartial, in normal form. Define a func. g on states by
$g(x) =$ smallest integer $i$ s.t. no follower $y$ of $x$ has $g(y) = i$.

\[ \text{follower} = \text{direct follower i.e.} \]
\[ x \rightarrow y \text{ is valid move} \]

**def** $\text{mex}(A) =$ minimal excluded integer $\geq 0$ from a set $A$

$\text{mex}(A) = \min(\mathbb{N} \setminus A)$

e.g. $\text{mex}(\{0, 1, 3, 4\}) = 2$

$\text{mex}(\{1, 3, 5\}) = 0$
\[ g(a) = \text{mex} \left( \{ g(c), g(b), g(d) \} \right) \]
\[ g(b) = \text{mex} \left( \{ g(f) \} \right) \]

e has no followers, so \[ g(e) = \text{mex} \left( \emptyset \right) = 0. \]
\[ g(f) = \text{mex}\{g(e)\} = \text{mex}\{0\} = 1 \]
\[ g(d) = \text{mex}\{g(f)\} = \text{mex}\{1\} = 0 \]
\[ g(c) = \text{mex}\{g(d), g(e), g(c)\} = \text{mex}\{0, 1, 0\} = 2 \]

\textbf{Thm} x is a P-position if and only if \( g(x) = 0 \)
**Example Subtraction** \( \{1, 2, 3\} \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( g(n) )</th>
<th>followers</th>
<th>( g(follens) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0, 1</td>
<td>0, 1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0, 1, 2</td>
<td>0, 1, 2</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1, 2, 3</td>
<td>1, 2, 3</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2, 3, 4</td>
<td>2, 3, 0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>3, 4, 5</td>
<td>3, 0, 1</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>4, 5, 6</td>
<td>0, 1, 2</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ g(n) = n \pmod{4} \]

\[ n \pmod{4} \]
can prove by induction.

e.g. sub. set \( \{1,3,4\} \)

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
2 & 0 & 1 & 1 \\
3 & 1 & 0.2 & 0.0 \\
4 & 2 & 0.1,3 & 0.1,1 \\
5 & 3 & 1.2,4 & 1.0,2 \\
6 & 2 & 2.3,5 & 0.1,3 \\
7 & 0 & 3.4,6 & 1.2,2 \\
8 & 1 & 4.5,7 & 2.3,0 \\
9 & 0 & 5.6,8 & 3.2,1 \\
10 & 1 & 6.7,9 & 2.0,0 \\
\end{array}
\]
Claim: pattern is 0101232 repeating.

\[ g(n) = \begin{cases} 
0 & n \equiv 0 \text{ or } 2 \mod 7 \\
1 & n \equiv 1 \text{ or } 3 \\
2 & n \equiv 4 \text{ or } 6 \\
3 & n \equiv 5 
\end{cases} \]

Proof: By induction. Checked for \( n \leq 10 \).

Assume true for all \( m < n \).

Case: \( n \equiv 0 \mod 7 \) then followers are \( n-1, n-3, n-4 \) are \( \equiv 6, 4, 3 \mod 7 \).
by ind. hyp. \( g(n-1) = 2 \)
\( g(n-3) = 2 \) and \( g(n-4) = 1 \)
so \( g(n) = \text{mex}\{2, 2, 1\} = 0 \)

Case \( n \equiv 1 \mod 7 \)
\[
\begin{align*}
\text{if} & \quad n \equiv 6 \mod 7 \\
\text{then once} & \quad k \text{ consecutive values repeat, pattern is found.}
\end{align*}
\]
e.g. \( A = \{1, 3, 4\} \), \( K = 4 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(n) )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

If 4 values repeat:

\[
\begin{align*}
abcde & \quad \cdots \quad abcde \\
\uparrow & \quad \uparrow & \quad \uparrow & \quad \uparrow \\
n-4 & \quad n-3 & \quad n-1 & \quad n
\end{align*}
\]

Then keep repeating.

\( g(n) \leq K \) so \( K \) consecutive values have \( \leq (K+1)^K \) options.
Thm. $x$ is a P-position if and only if $g(x)=0$

pf. Let $M(x) = \text{max number of moves starting at } x$.
Prove claim by induction on $M(x)$.
$M(x) = 0$ normal play so $x$ is p-pos. and $g(x) = 0$.
If true for $M(y) < n$ and $M(x) = n$. 
Followers of $x$ have $M(y) < n$ so ind. hyp. applies to them.

so $x$ is $N$-pos. $\iff$

some follower is $P$-pos.

$\iff$ some follower has $g(y) = 0$

$\iff$ $g(x) \neq 0$

Alternatively: $x$ is $P$-pos.

$\iff$ all foll. are $N$-pos.

$\iff$ all foll. have $g(y) \neq 0$

$\iff$ $g(x) = 0$. 
from a: ≤ 4 moves to end.
Grundy's Game

States: several piles of coins
Moves: split one pile into two unequal piles
Normal play: last move wins

E.g. start: 17

1, 16
1, 6, 10
11, 5, 10
11, 23, 10
11, 2, 12, 10
11, 22, 12, 10
11, 22, 12, 11, 12
Qn: How to win?

Note: the game with several piles is a disjunctive sum of games with 1 pile.

Thm: For position \((x, y)\) in \(G_1 + G_2\),

\[ g(x, y) = g(x) \oplus g(y) \]

here,

\[ g(x, y) = \text{value of pos } (x, y) \]

in the sum \(G_1 + G_2\),
e.g.

\[ g(a) = 1 \quad g(b) = 0 \]

so \[ g(a, b) = 1 \oplus 0 = 1 \]

winning moves \((a, b) \rightarrow (b, b) \rightarrow (a, f)\)

Qn: what moves win from \((b, c)\)?

Sol.: \((b, d)\) or \((b, e)\).
<table>
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<tr>
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<th>( \text{foll.} )</th>
<th>( g(\text{foll.}) )</th>
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<tr>
<td>1</td>
<td>0</td>
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<td>0</td>
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<td>2</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>(1,2)</td>
<td>( g(1) \oplus g(2) = 0 \oplus 0 = 0 )</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>(1,3)</td>
<td>( g(1,3) = g(1) \oplus g(3) = 1 )</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>(1,4)</td>
<td>( g(1,4) = 0 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2,3)</td>
<td>( g(2,3) = 1 )</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>(1,5)</td>
<td>0 \oplus 2 = 2</td>
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e.g. $g(8) = \text{mex} \{g(1, 7), g(2, 6), g(3, 5)\}$

$= \text{mex} \{g(1) \oplus g(7), g(2) \oplus g(6), g(3) \oplus g(5)\}$

$= \text{mex} \{0 \oplus 0, 0 \oplus 1, 1 \oplus 2\} = 2$

No formula is known.

Plot of first $2^{14}$ values, and $\sqrt{n}$. 
Thm In a sum of games \( G_1 + \ldots + G_k \), the Sprague-Grundy value of \((x_1, \ldots, x_k)\) is the NIM-sum of the values in the separate games:

\[
g(x_1, \ldots, x_k) = g(x_1) \oplus \cdots \oplus g(x_k)
\]

e.g. NIM with 1 pile has

\[
g(n) = n \quad \text{[induction]}
\]

so \( g(n_1, \ldots, n_k) = n_1 \oplus \cdots \oplus n_k \)

and \( p\text{-pos.} \iff \text{NIM-sum is 0} \).
e.g. In Grundy's game,
\[ g(3,4,9) = g(3) \oplus g(4) \oplus g(9) = 0 \oplus 0 \oplus 0 = 0 \]
so \((3,4,9)\) is a \(p\)-position.

**Proof** By induction on
\[ M(x_1, \ldots, x_k) = M(x_1) + \ldots + M(x_k). \]

If no moves: \( g_i(x_i) = 0 \)
and also \( g(x_1, \ldots, x_k) = 0 \).

**Claim 1:** If \((y_1, \ldots, y_k)\)
is a follower of \((x_1, \ldots, x_k)\)
then \( g_1(y_1) \oplus \ldots \oplus g_k(y_k) \)
\[
\neq g_k(x_1) \oplus \ldots \oplus g_k(x_k).
\]

Claim 2: If \( m < g_1(x_1) \oplus \ldots \oplus g_k(x_k) \)
then some follower has
\( g(y_1) \oplus \ldots \oplus g(y_k) = m. \)

Given Claims 1, 2 the theorem follows.

Proof of Claim 1:
A follower has \( y_i = x_i \).
except for a single $j$

where $x_j \rightarrow y_j$

$g_j(y_j) \neq g_j(x_j)$ so adding

$g_i(x_i) = g_i(y_i)$ for $i \neq j$

proves the claim.

**Proof of claim 2:** Let

$n_i = g_i(x_i)$ and $m < n$

$n = n_1 \oplus n_2 \oplus \ldots \oplus n_k$

we show that can replace
a single \( n_j \) by \( n'_j < n_j \)

s.t. \( n_0 - \cdots - n_j = m \)

\( X_j \) has \( g_j(x_j) = n_j \).

Some follower \( X_j \rightarrow Y_j \) has \( g_j(y_j) = n_j \).

need \( n'_j = n_j \oplus m \oplus n \)

For some \( j \) \( n'_j < n \) as in \( \text{NIM} \).
\[
\begin{array}{cccc}
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 \\
\end{array}
\]
\[
\begin{array}{c}
\text{n}_i \\
\text{n} \\
\text{m} \\
\end{array}
\]
\[
\uparrow
\]
\[
\text{n} \text{ has } 1, \quad \text{m} \oplus \text{n} \text{ has a } 1
\]
\[
\text{m} \text{ has } 0
\]
some \( n_j \) has a 1 in that coordinate,
so \( n_j \oplus m\oplus n < n_j \)
as needed \( \square \)
Corollary: In a game $G$, 
$g(x) = n$ if and only if $(x, n)$ is a P-position in $G + \text{NIM}$.

Misere games are hard.

Partizan games are fascinating.

References:

[On Numbers and Games]

[Winning Ways]