O-som games

- 2 player games
- Fully adversarial: One player's gain is the other's loss.

Strategy is a rule giving all your decisions in a game (in every possible situation).

E.g., SUBTRACTION {1, 2} start with 4.

Player 2 has 4 strategies: If P1 took 1: take 1 or 2
If P1 took 2: take 1 or 2

Player 1 has more, e.g.

Take 1, then if P2 takes 1 take 2
if P2 takes 2 take 1

(can also allow random strategies)

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Matrix form of a game.

Rows = strat. of P1
Cals. = strat. of P2.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & -1 & 1 & 1 \\
-1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

If P1 chooses strat. \( i \) and P2 strategy \( j \), then P2 pays \( A_{ij} \) to P1 (\( A_{ij} \) can be negative).

e.g. \( A_{ij} = 1 \) if P1 wins
\( A_{ij} = -1 \) if P2 wins.
In this case, winning strat. for P1 is a row with all +1.

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

Winn. strat for P2: column of -1's

In a combinatorial game one of these must exist.

\(O\)-sum. games might not have this.

e.g. \[
\begin{pmatrix}
-1 & +1 \\
+1 & -1
\end{pmatrix}
\] matching pennies
odd/even
0-sum game is given by a matrix $A_{n \times m}$

$P_1$ picks one of $n$ rows \{ simultaneous \}
$P_2$ picks one of $m$ cols

$P_1$ gains $A_{ij}$, $P_2$ pays $A_{ij}$.

Total gain is always 0.

$$\begin{pmatrix} 2 & 3 & 7 \\ 4 & -9 & 11 \end{pmatrix}$$

Pure strategy: pick one row.
Mixed strategy: pick randomly.

∞ game: bigger number:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 0 & -1 & -1 & -1 & -1 \\ 2 & 1 & 0 & -1 & -1 & -1 \\ 3 & 1 & 1 & 0 & -1 & -1 \\ 4 & 1 & 1 & 1 & 0 & -1 \\ 5 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$
e.g.  Rock - Paper - Scissors

\[
\begin{pmatrix}
R & P & S \\
R & 0 & -1 & 1 \\
P & 1 & 0 & -1 \\
S & -1 & 1 & 0 \\
\end{pmatrix}
\]

No way row avoids risk of losing.

Loonie - Twonie game:

\[
\begin{pmatrix}
1 & 0 \\
0 & 2 \\
\end{pmatrix}
\]

P1 can be sure to get \( \geq 0 \) (either row)
P2 can be sure to pay \( \leq 1 \) (col. 1)

Can play randomly!

Let P1 use a coin toss to pick.

P1's expected gain is \( \frac{1}{2}(1) + \frac{1}{2}(0) \) in col. 1
\( \frac{1}{2}(0) + \frac{1}{2}(2) \) in col. 2

\( \left( \frac{1}{2}, 1 \right) \)
For \( p_1 \): Each row might get 0

Pick \( \frac{1}{2} - \frac{1}{2} \) then get \( \geq \frac{1}{2} \) no matter what \( p_2 \) chooses.

Pick row 1 w.p. \( \frac{2}{3} \)

2 w.p. \( \frac{1}{3} \).

\[
\begin{pmatrix}
\frac{2}{3} & \frac{1}{3}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix}
= \begin{pmatrix}
\frac{2}{3} & \frac{2}{3}
\end{pmatrix}
\]

Guarantees \( \geq \frac{2}{3} \)

Player 2 can limit avg. loss to \( 2/3 \) by picking

Col. 1 w.p. \( \frac{2}{3} \)

Col. 2 w.p. \( \frac{1}{3} \)

\[
\begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
\frac{2}{3} \\
\frac{1}{3}
\end{pmatrix}
= \begin{pmatrix}
\frac{2}{3} \\
\frac{2}{3}
\end{pmatrix}
\]

safety strat.
Safety strategy and value

For Pl: maximize the worst outcome.

For each row consider the min. value, take max of these.

Same for each distribution for random row.

\[ X = (x_1, \ldots, x_n) : \text{pick row } i \text{ w.p. } x_i \]

Consider the min \((xA)\) and pick \(x\) which maximizes that.
Recall: Safety value: what a player can guarantee getting

Safety strat.: any strat. that achieves the safety value

If restricted to pure strat:

P.1. can achieve $\geq \max_i \min_j A_{ij}$  \hspace{1cm} (at least)

P.2. can achieve $\min_i \max_j A_{ij}$  \hspace{1cm} (at most)

Thm: For any $A$, $\max_i \min_j A_{ij} \leq \min_i \max_j A_{ij}$

Proof: Let $i^*$ be any $i$ that gives $\max_j A_{i^*j}$ on LHS

Let $j^*$ be any $j$ that gives $\min_i A_{ij}$ on RHS

$\max_i \min_j A_{ij} = \min_i A_{i^*j^*} \leq A_{i^*j^*} \leq \max_i A_{i^*j^*} = \min_i \max_j A_{ij}$
safest pure strat. is row 4

P.I safety value \( \geq 0 \)

Can do better e.g. \((0, 0, 0, \frac{1}{2}, \frac{1}{2})\) gives \(\frac{113}{2} > 0\)

If LHS = RHS : \(A_{i,j,k}\) is minimal in its row \} saddle
and maximal in its col. \} point.

**Thm (von Neumann’s minimax):** For any \(A\),

\[
\max_{x \in \Delta^n} \min_{y \in \Delta^m} x^T A y = \min_{y \in \Delta^m} \max_{x \in \Delta^n} x^T A y
\]

i.e. the safety valves are equal. This is the value of \(A\) as a 0-sum game.
\( \Delta^n \) : space of mixed strategies on \( n \) rows.

\( \Delta^n = \{ x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i, \quad \sum x_i = 1 \} \)

\( n=2 \):

\( n=3 \):

\((0,0,1) \rightarrow x_3 \)

\((1,0,0) \rightarrow x_2 \)

\((0,1,0) \rightarrow x_1 \)
E.g. \[ A = \begin{pmatrix} 0 & 3 & -2 & 4 \\ 4 & 1 & 2 & 0 \end{pmatrix} \]

Find P.I. safety strat.

Given \[ x^T = (x_1, x_2) \] what is \[ \min_y x^T A y \] ?

\[ x^T A = (4x_2, 3x_1 + x_2, -2x_1 + 2x_2, 4x_1) \]
\[ = (4x_2, 1 + 2x_1, 2 - 4x_1, 4x_1) \]

\[ \min_{y \in \Delta_m} x^T A y = \min_y x^T A y \text{ pure} \]

\[ \min_y x^T A y = \min \{ 4 - 4x_1, 1 + 2x_1, 2 - 4x_1, 4x_1 \} \]

Using \[ x_2 = 1 - x_1 \]
optimal \( x: 4x_1 = 2 - 4x_1 \implies x_1 = \frac{1}{4} \)

\[ x^T A = \begin{pmatrix} 3 & 0.5 & 1 & 1 \end{pmatrix} \]

Safety value is 1.

For \( p_2 \): given \( y \in \Delta^4 \), get

\[ \max \left( 3y_2 - 2y_3 + 4y_4, 4y_1 + y_2 + 2y_3 \right) \]

Using minimax thm: know this is \( = 1 \) for the safety strat.

i.e. need \( y \) s.t.

\[ 3y_2 - 2y_3 + 4y_4 \leq 1 \]

\[ 4y_1 + y_2 + 2y_3 \leq 1 \]

If \( p_1 \) uses \( (\frac{1}{4}, \frac{3}{4}) \) safety strat, then \( p_2 \) should only use col. 3 and 4.
-2y_3 + 4y_4 \leq 1 \quad \Rightarrow \quad y_3 = \frac{1}{2}
\quad 2y_3 \leq 1 \quad \quad \quad \quad \quad y_4 = \frac{1}{2}
\quad y_3 + y_4 = 1

Safety strat for P-2 is \( y^* = (0, 0, \frac{1}{2}, \frac{1}{2}) \)

\[ Ay = \begin{pmatrix} 0 & 3 & -2 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ y_3 \\ y_4 \end{pmatrix} = (1) \]

Often the optimal strat. equalize payoff

not always, e.g. S.P.

\[ \begin{pmatrix} 0 & 0 & 2 & 0 \\ 4 & 5 & 3 & 6 \end{pmatrix} \]

In 2x2 game, either \{ S.P. equalized payoff \} pure optimal strat. is optimal
Recall: **Optimal reply**: A player's best response to a known strategy by opponent. (Any such reply if several are equally good)

\[ A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 5 & 1 \\ 6 & 0 & 4 \end{pmatrix} \]

**Q.** If \( y = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \), optimal reply for P.1. is?

**A.** P.1's payoff vector is \(Ay = \begin{pmatrix} 8/3 \\ 8/3 \\ 10/3 \end{pmatrix}\). Optimal reply is row 3, \( x^T = (0, 0, 1) \).

**Q.** Same for \( A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{pmatrix} \) same \( y \).

**A.** \( Ay = \begin{pmatrix} 8/3 \\ 8/3 \\ 10/3 \end{pmatrix} \) so optimal reply is \((p, 1-p, 0)^T\) for any \( p \).
Similarly, payoff vector for $p_2$ is $x^\top A$

optimal reply chooses among minimizing entries of $x^\top A$.

**Nash Equilibrium**

A N.E. is a pair of strategies $(x^*, y^*)$ such that $x^*$ is an optimal reply to $y^*$ and $y^*$ is optimal reply to $x^*$.

In a 0-sum game, $x^*$ is optimal against $y^*$ if there is some $V$ s.t. entries of $Ay^*$ all $\leq V$ and if $x_i^* \neq 0$ then $(Ay^*)_i = V$.

Similarly, $y^*$ optimal against $x^*$ if $\forall V$ s.t. $(x^\top A)_j \geq V$ and if $y_j^* \neq 0$ then $(x^\top A)_j = V$.
For any \( A \) there is a \( v = \text{Val}(A) \) and some \( x^*, y^* \) s.t. entries of \( x^T A \) are \( \geq v \) \( \Rightarrow \{ \frac{x^T A y^*}{y^*} \leq v \) entries of \( A y^* \) are \( \leq v \) so \( x^T A y^* = v \).

This implies \( (y^*)_i \neq 0 \) only where \( x^T A = v \) and \( (x^*)_i \neq 0 \) only where \( A y^* = v \).

So \( (x^*, y^*) \) are a Nash Equilibrium.

E.g. \( A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 1 & 3 \\ 0 & 3 & 3 \end{pmatrix} \) \( x^T = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \) \( y = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} \) \( x^T A = (\frac{3}{2}, \frac{3}{2}, \frac{3}{2}) \) y optimal against \( x \). \( A y = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} \) x optimal against y. \( (x, y) \) is a N.E. \( \text{Val}(A) = \frac{3}{2} \).
If \( y = \left( \frac{1}{2}, 0 \right) \) \( y \) is optimal against \( x = \left( \frac{1}{4}, \frac{3}{4} \right) \).

\( A y = \left( \frac{1}{2} \right) \) \( x \) not optimal against \( y \).

**Domination** strategy \( i \) dominates strategy \( k \) for \( P1 \)

if for all \( j \), \( A_{ij} \geq A_{kj} \).

strict domination: \( A_{ij} > A_{kj} \quad \forall j \)

For \( P2 \): \( j \) dominates \( k \) if \( \forall i \) \( A_{ij} \leq A_{ik} \)

**Claim:** If strat. \( i \) dominates strat. \( k \) for \( P1 \), then replacing \( k \) by \( i \) cannot harm \( P1 \).
Lemma: removing dominated strategies does not change the value of a game.

e.g. \[ A = \begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 2 & 0 \\ 4 & 1 & 0 & 5 \end{pmatrix} \] rows 2, 3 dominated by row 1

reduces to \[ A' = \begin{pmatrix} 1 & 2 & 4 & 6 \end{pmatrix} \] col. 4 dominated by col. 1.

\[ A'' = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & 0 \end{pmatrix} \]

note: if end up with 1x1 matrix, that's a saddle point in original game.

\[ Ay = (v, v) \] since optimal \( x \) uses both rows

optimal \( x \) has \( x^T A = (v, v, > v) \)

optimal \( y \) is \( (y_1, y_2, 0)^T \).
Proof: Let $v = \text{Val}(A)$. Assume row $i$ dominates row $k$. Optimal $x$ has $x^TA$ has entries $\geq v$.

Let $\hat{x}$ be $x$ with weight transferred from $k$ to $i$.

E.g. $x^T = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, $i = 2$, $k = 3$, $\hat{x}^T = (\frac{1}{4}, \frac{1}{4}, 0)$

$\hat{x} = x + x_k(0 \ 0 \ x_0 - x_0)$

so $\hat{x}^T A = x^T A + x_k(\text{row } i - \text{row } k)$

$\geq 0$ entries

so entries of $\hat{x}^T A$ are $\geq v$.

1. $p_1$ can achieve $\geq v$ without using row $k$.

2. $p_2$ can achieve $\leq v$.

so $\text{Val}(A') = v$. $\square$
Methods for solving 0-sum games

Solve: find the value and an optimal strategy for each player.

Optimal $x$ for P.1 is any $x$ s.t. entries of $x^t A \geq v$

" $y $ " for P.2 $y$ $A y \leq v$

e.g. \[ A = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 4 & 3 \end{pmatrix} \] \[ x = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \] for P.1 guarantees $\geq 2$

\[ x^t A = (2, 2, 3) \]

\[ y = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \] for P.2 \[ A y = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \]

There might be other optimal $x$ or $y$.

$A \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ so $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ not optimal.
Methods
- Saddle point: pure strategies are optimal.
- Simplify by
do:
- Dominated strategies
- $2 \times m$ or $n \times 2$ games:
- Symmetries.
- General algorithms: Assume $x, y$ are optimal = N.E.
  If we know which $x_i \neq 0$ and which $y_j \neq 0$ can find $x, y$ by solving lin. eqns.
e.g. \( A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \) if \( x = (x, 1-x) \) \( \min (x^T A) = \)

\( \min x^T A \) is maximized for any \( x \in [\frac{1}{4}, \frac{3}{4}] \)

so value is 1, multiple opt. strategies for p1.

Symmetries: between strategies or players.

e.g. \( A_{ij} = |i - j| \quad i, j \in \{0, 1, 2, \ldots, n\} \)

e.g. (Book): 

\[
\begin{array}{ccc}
\square & \square & \square \\
\end{array}
\]

submarine + bomber

Symmetry: sub has 2 strat: 

\[
\begin{array}{ccc}
\square & \square & \square \\
\end{array}
\]

reduced 9x2 to 3x2.
Symmetry of players:

\[
\begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{pmatrix}
\]

Swapping p₁ and p₂ means replacing A by \(-A^T\).

If \( A = -A^T \) then swapping player roles makes no diff.

Then Val must be 0.
\((A y)_i\) must have same value \(v\) when \(x_i \neq 0\) \(\leq \) \(k\) eqn.

\((x^TA)_j\) \(\leq \) \(v\) \(\leq \) \(y_j = 0\) \(\leq \) \(l\) eqn.

1+K+l variable for \(x, y, v\)

K+l+2 eqns: \(\sum x_i = 1\) \(\sum y_j = 1\) (one is redundant)

Sol. is a N.E. iff 
other rows of \(Ay\) are \(\leq v\)

\[ + x_i \in \Delta^x \quad y_j \in \Delta^y \]
other cols of \(x^TA\) are \(\geq v\).

Algor.: "Guess" which rows and cols are used in optimal \(x, y\). \((2^n-1)(2^m-1)\) possibilities.

For each option solve lin. eqn. and check \((x)\)
Special case: all rows + cols are used. Equalizing payoffs

\[ A y \text{ must be } \begin{pmatrix} v \\ v \\ \vdots \\ v \end{pmatrix} \]

\[ x^T A = (v, -v) \]

E.g. \[ A = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 4 & 3 \end{pmatrix} \]

\[ \begin{pmatrix} 4x_1 + 4x_2 + 3(x_1 + x_2) \end{pmatrix} \]

\[ 4x_1 = 4x_2 = 3(x_1 + x_2) \]

\[ x_1 + x_2 = 1 \]

So no sol.

E.g. \[ A = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix} \]

Suppose for some A get \[ x = \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{6} \right) \]

\[ y = \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix} \]

Not a N.E since \[ y_i \geq 0 \] and \[ x_i \geq 0 \]

If \( x \in \Delta^m \) \[ y \in \Delta^m \] and \( x^T A, A y \) are const. vectors

Then \( x, y \) is a solution.
Simplex algorithm

Solves linear optimization

Variables \((x_i)\)

\[\text{find } \max \sum u_i x_i\]

\[\text{given } \sum c_{ij} x_i \leq b_j \text{ for all } j\]

\[C x \leq B \text{ for matrix } C, \text{ vector } B\]

Finding N.E in O sum game can be written as

lin. optimization problem.

Variables are \((x_i) (y_j) v\).

Constraints: \(x_i \geq 0, y_j \geq 0, \sum x_i = 1, \sum y_j = 1\)

for all \(i \sum A_{ij} y_j \leq v, \sum x_i A_{ij} \geq v \text{ for all } j\)
If $A = A^T$ e.g. $A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, then Value $= 0$ and use same strat.

Note: Convex combination of $v_1, \ldots, v_k$ is $\sum a_i v_i$ where $a \in \Delta^k$ $\left[ a_i \geq 0 \right.$ $\sum a_i = 1 \right.$.

For $x \leq \Delta^m$ and $A$ a $m \times n$ matrix, $x^T A$ is a convex combination of rows of $A$.

Claim: adding $x^T A$ as a row to $A$ does not change the value.
If a row is dominated by a combination of other rows, it can also be removed.

\[
\begin{pmatrix}
2 & 0 \\
0 & 4 \\
1 & 1 \\
\end{pmatrix} 
\]

Row 3 \leq \frac{1}{2}(\text{row 1}) + \frac{1}{2}(\text{row 2})

\[
\begin{pmatrix}
2 & 0 & 1.5 \\
0 & 4 & 2 \\
1 & 1 & 2 \\
\end{pmatrix} 
\]

Col 3 \geq \frac{1}{2}(\text{col 1}) + \frac{1}{2}(\text{col 2})

\text{domination}
Def: A convex set is a set $K$ s.t. $\forall x, y \in K$ the segment $\overline{xy} \subset K$

i.e. $z = tx + (1-t)y \in K$ for all $t \in [0,1]$

e.g. $\mathbb{R}^n$
Hyperplane separation Lemma

Def: A set $K \subset \mathbb{R}^n$ is closed if it includes its boundary.

Lemma: Assume $K \subset \mathbb{R}^n$ closed and convex, and $0 \notin K$, then there are $c > 0$ and $z \in \mathbb{R}^n$ such that $z \cdot v > c$ for all $v \in K$.

Note: $z \cdot v = \langle z, v \rangle = z^T v$
Proof: Let \( z \) be the point in \( K \) closest to \( 0 \).

Such \( z \) exists (\( K \) closed)
\( z \) is unique (\( K \) convex)
If two candidates have the same distance,
then \( \frac{z + z'}{2} \in K \) is closer to \( 0 \).

Claim: \( z \cdot v \geq z \cdot z = ||z||^2 \)
So result holds with \( c = \frac{1}{2} ||z||^2 \)
the segment \( z \) has points
\( v \cdot t + z(1-t) = z + t(v-z) \) for \( t \in [0,1] \)
\( z + t(v-z) \in K \)
\( z \) nearest to 0 so \( z + t(v - z) \) farther.

For \( t \in [0, 1] \) get

\[
(z + t(v - z))(z + t(v - z)) \geq z \cdot z
\]

\[
z \cdot z + 2z \cdot t(v - z) + t^2(v - z)(v - z) \geq z \cdot z
\]

\[
2z \cdot t(v - z) + t^2(v - z)(v - z) \geq 0 \quad \text{for} \quad t \in [0, 1]
\]

so \( \geq 0 \)

\[
z \cdot v - z \cdot z \geq 0 \quad \text{so} \quad z \cdot v \geq z \cdot z
\]

\[
\text{Let } r, \text{ is}
\]

\[
2z(v - z)
\]

\[
0
\]

\[
t
\]
Thm! A is mxn matrix then

\[
\max_{x \in \Delta^m} \min_{y \in \Delta^n} x^T A y = \min_{y \in \Delta^n} \max_{x \in \Delta^m} x^T A y.
\]

Lemma! If A, B are convex, closed, bounded sets in \( \mathbb{R}^m, \mathbb{R}^n \)

\[ f : A \times B \rightarrow \mathbb{R} \text{ is contin. then} \]

\[
\max_{x \in A} \min_{y \in B} f(x, y) \leq \min_{y \in B} \max_{x \in A} f(x, y)
\]

[use this for \( f(x, y) = x^T A y \) \( f \) is cont. on \( \Delta^m \times \Delta^n \).]
\[ \min_{y \in B} f(x, y) \leq f(x, y_0) \leq \max_{x \in A} f(x, y_0) \]

for all \( x, y_0 \) so also

\[ \max_{x \in A} \min_{y \in B} f(x_0, y) \leq \min_{y \in B} \max_{x \in A} f(x, y_0) \]

So

\[ \max_{x \in A} \min_{y \in B} f(x_0, y) \leq \min_{y \in B} \max_{x \in A} f(x, y_0) \]

This gives one direction in minimax thm: \( \text{LHS} \leq \text{RHS} \)

Note! need \( f \) cont. and \( A, B \) closed + bounded to guarantee that the min and max are attained.
Idea for $\geq$: Assume RHS is $\geq \lambda$ for some $\lambda$.

Deduce that LHS $\geq \lambda$.

This implies LHS $\geq$ RHS.
Midterm next week:

Everything up to (incl.) 0-sum games.

Sample later today.

Recall: Goal: A matrix A

\[
\max_{x \in \Delta^m} \min_{y \in \Delta^n} x^T A y = \min_{y \in \Delta^n} \max_{x \in \Delta^m} x^T A y
\]

Seen: \( \max \min \leq \min \max \)

Proof od \( \geq \) show that cannot have

\[
\max_{x} \min_{y} x^T A y < \lambda < \min_{y} \max_{x} x^T A y
\]

for any \( \lambda \).
If $\hat{A}_{ij} = A_{ij} + c$ for all $ij$ then

$x^T\hat{A}y = x^TAy + c$ for mixed $x,y$.

If (x) holds then $\hat{A}_{ij} = A_{ij} - \lambda$ has

$$\max \min x^T\hat{A}y < 0 < \min \max y^T\hat{A}x$$

So enough to rule out (x)

Build a convex closed set $K$ from $A$:

$$K = \{ Ay + v : y \in \Delta^n, v_i \geq 0 \text{ for all } i \} \subset \mathbb{R}^m$$
e.g. $A = \begin{pmatrix} 3 & 4 & 2 & 0 \\ 1 & 0 & 3 & 6 \end{pmatrix}$

Claim: If $0 < \min \max_y x^T Ay$ then $0$ not in $K$.

call $\alpha = \min \max_y x^T Ay$

then for all $y$, some coord. of $Ay$ is $\geq \alpha$.

so some coord. of $Ay + v \geq \alpha$.

so $Ay + v = 0$ impossible.

$K = \{ Ay + v : y \in \Delta^n \}$
Hyperplane sep.: \( \exists z \text{ s.t. } \forall u \in K \quad z \cdot u > c > 0 \)

Claim: coord. \( z_i > 0 \).

Proof: if \( z_i < 0 \) then take any \( u \in K \), add \( L \) to \( u \). This is still in \( K \).

\[ z \cdot \hat{u} = z \cdot u + L \cdot z_i \quad \text{if } z_i < 0 \quad \text{then } z \cdot \hat{u} < 0 \]

for \( L \) large enough.

Can replace \( z \) by \( z/\varepsilon z_i \) to get \( z \in \Delta^m \).

\( z \) is strat. for \( p_1 \) s.t. \( \forall y \in \Delta^m \) and \( v \geq 0 \),

\[ z \cdot (Ay + v) \geq c > 0 \]

If \( v = 0 \) get \( z^TAy > 0 \)
So \( \min_y z^T Ay \geq 0 \).

So also \( \max_x \min_y x^T Ay \geq \min_y z^T Ay \geq 0 \).

\[
\text{Seen: } \min_y \max_x x^T Ay > 0 \text{ then } \max_x \min_y x^T Ay \geq 0
\]

Note: In \( \infty \) dim. Hyperplane sep only gives \( > 0 \).

The thm still holds if \( A \) is contin. on compact space.

E.g. players pick \( u, v \in [0, 1] \) payoff is \( A(u, v) \).