General Sum Games

Each player picks a strategy, e.g., i, j.
Outcome is \((a_{ij}, b_{ij})\) for P1 and P2.

Special case: \(b_{ij} = -a_{ij}\) \([0\text{-sum game}]\)

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<thead>
<tr>
<th></th>
<th>Bond</th>
<th>Venom</th>
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</thead>
<tbody>
<tr>
<td>Bond</td>
<td>((5, 3))</td>
<td>((2, 2))</td>
</tr>
<tr>
<td>Venom</td>
<td>((-1, -1))</td>
<td>((3, 5))</td>
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Note: For now assume no communication.

For P1 Payoff matrix \(A = \begin{pmatrix} 5 & 2 \\ -1 & 3 \end{pmatrix} \)
General sum games (2 players)

P1 has \( m \) actions \( \) each pair \( ij \) results in payoff \( a_{ij} \)

P2 has \( n \) actions

E.g.

\[
\begin{array}{cc}
(1, 0) & (2, 1) \\
(0, 3) & (3, 2)
\end{array}
\]

Optimal reply to row 1 \( \) col 2

Row 2 \( \) col 1

col 1 \( \) row 1

col 2 \( \) row 2

\[
A = \begin{pmatrix}
1 & 2 \\
0 & 3
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
0 & 1 \\
3 & 2
\end{pmatrix}
\]

No stable pair of pure strategies.

If P1 picks row w.p. \( \frac{1}{2}, \frac{1}{2} \) = \( x^T \)

P2 payoff vector is \( x^T B = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \end{pmatrix} \)

Any strategy, pure or mixed is an optimal reply.
War and Peace

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<tr>
<th></th>
<th>war</th>
<th>Peace</th>
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<tr>
<td>war</td>
<td>(-2, -2)</td>
<td>(-1, -3)</td>
</tr>
<tr>
<td>Peace</td>
<td>(-3, -1)</td>
<td>(1, 1)</td>
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Optimal reply to war is war.
Optimal reply to peace is peace.

Def: A Nash Equilibrium is pair of strategies \( x, y \) such that \( x \) is an optimal reply to \( y \) and vice versa.

E.g.: (war, war) or (peace, peace)

Note: A general sum game can have multiple N.E. with different outcomes.

In 0-sum game, all have same value.
only N.E. is \( x^* = \left( \frac{1}{2}, \frac{1}{2} \right) \quad y^* = \left( \frac{1}{2}, \frac{1}{2} \right) \)

In WTP take \( x^* = \left( \frac{2}{3}, \frac{1}{3} \right) \quad y^* = \left( \frac{2}{3}, 1 \right) \)

\[
x^T B = \left( \frac{2}{3}, \frac{1}{3} \right) \begin{pmatrix} -2 & -3 \\ -1 & 1 \end{pmatrix} = \left( -\frac{5}{3}, -\frac{5}{3} \right)
\]

Each optimal against the other.

\[
A y = \begin{pmatrix} -\frac{5}{3} \\ -\frac{5}{3} \end{pmatrix}
\]

**Theorem:** Any finite game has at least one N.E.

First game has one.

WTP: 3 N.E. \[ \begin{cases} \text{(War, War)} \\ \text{(Peace, Peace)} \end{cases} \]

\( (x, y) \quad x = y = \left( \frac{2}{3}, \frac{1}{3} \right) \)
Safety values + Strategies

Safety value for P1 is best they can get against any opponent.

This is the value of A as a 0-sum game.

\[
\max_x \min_y x^T Ay
\]

E.g. \(A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\) so safety value is 1
safety strat. is (row 1)

For P2: Safety value is \(\max_y \min_y x^T By = \max_y \min_y y^T B^T x = y^T B^T x\)

This is the value of \(B^T\) as a 0-sum game.

\(-= Value (-B) as 0-sum game.\)
Exercise: Find safety values for War + Peace

\[ A = \begin{pmatrix} -2 & -1 \\ -3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -3 \\ -1 & 1 \end{pmatrix} \]

In \( A \), 2 is a s.p. so safety value is -2
safety strat is War

Same for \( P \).

Claim: If \((x, y)\) is a N.E. and safety values are
\((V_1, V_2)\), then \(x^T Ay \geq V_1, \quad x^T By \geq V_2\)

Proof: If \(x^T Ay < V_1\), then P1's safety strategy would be
better than \(x\) against \(y\), so \(x\) is not optimal reply
to \(y\), so \((xy)\) is not a N.E.
Prisoner's Dilemma

Each prisoner can cooperate with the other or defect (testify).

\[
\begin{array}{cc}
\text{C} & \text{D} \\
\text{C} & (-2, -2) & (-6, -1) \\
\text{D} & (-1, -6) & (-3, -3) \\
\end{array}
\]

\[
A = \begin{pmatrix}
-2 & -6 \\
-1 & -3
\end{pmatrix}
\]

Optimal reply to anything is D.
Recall: A Nash Equilibrium is a pair of strategies, \((x, y)\) s.t. \(x\) is optimal against \(y\) and \(y\) optimal against \(x\).

Notes: Same holds for more than 2 players.

Strategy for each player optimal given all others.

For 0-sum games, same as safety/optimal strategy (Minimax), Not same in general.

### Prisoners' Dilemma

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<td>((0, -10))</td>
</tr>
<tr>
<td>Defect</td>
<td>((-10, 0))</td>
<td>((-8, -8))</td>
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<td>D</td>
<td>((0, -10))</td>
<td>((-8, -8))</td>
</tr>
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For P1 \( A = \begin{pmatrix} -2 & -10 \\ 0 & -8 \end{pmatrix} \) row 2 (D) dominates row 1 (C)

Claim: If a row is strictly dominated by another row in A then it not used in a N.E.
Similarly for dominated cols in B for P2

P.2 : \( B = \begin{pmatrix} -2 & 0 \\ -10 & -8 \end{pmatrix} \)

By domination, only N.E is (P, D).

(From weakly dominated strategies can be used in N.E.)

\[ A = \begin{pmatrix} 10 & 1 & 1 \\ 10 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix} \]

(row 1, col 1) is a N.E.
Tragedy of the commons

Several farmers. Each picks size of a herd $S_i$. Given these sizes, value of each unit is $(100 - \Sigma s_i)^+$

$x^+ = \max(x, 0)$

So payoff for player $i$ is $S_i(100 - \Sigma s_i)^+$

Look for pure N.E.

For $P_1$ get 0 if $S_i = 0$ or $S_i = 100 - \Sigma s_j \quad j \neq i$

So max payoff (optimal reply) is $S_i = \frac{1}{2}(100 - \Sigma s_j) \quad j \neq i$

$2S_i = 100 - \Sigma s_j \quad j \neq i$

$S_i = 100 - \Sigma s_j \quad \text{RHS same for all } i$
In a N.E this holds for every $i$, so $\forall i$

\[ s_i = 100 - \sum_0^{n-1} s_j \] so all $s_i$ are equal.

If there are $n$ farmers, $s_i = 100 - ns_i$

\[ s_i = \frac{100}{n+1} \]

Payoff for each is $\frac{100}{n+1} \left(100 - n \frac{100}{n+1}\right) = \left(\frac{100}{n+1}\right)^2$

Socially optimal strategy has $\sum s_i = 50$. Each $s_i = \frac{100}{2n}$

Payoff for each player is $\frac{100^2}{4n} > \frac{100^2}{(n+1)^2}$
Finding N.E. in general sum games

Pure N.E.: look for row i & col j optimal against each other.

Remove strictly dominated col./row. (iterate)

General method: e.g.

<table>
<thead>
<tr>
<th></th>
<th>(0, 0)</th>
<th>(5, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(6, 2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 1)</td>
<td>(3, 3)</td>
<td>(4, 0)</td>
</tr>
<tr>
<td>(2, 0)</td>
<td>(1, 5)</td>
<td>(6, 5)</td>
</tr>
</tbody>
</table>

(row 3, col 3) is a pure N.E.
(row 2, col 2)

Guess which rows + cols are used in (x, y) with prob ≠ 0.
E.g. rows {1, 2} cols {1, 2}
\[ x^* = (x_1, x_2, 0) \quad y^* = (y_1, y_2, 0) \]

\[ y \text{ optimal against } x \implies x^T B = (v, v, \leq v) \]

\[ B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ 0 & 2 \end{pmatrix} \quad \text{find} \quad x^* = \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \]

\[ x^T B = \left( \frac{3}{2}, \frac{3}{2}, 1 \right) \]

Similarly \[ A y = \begin{pmatrix} v \\ v \\ \leq v \end{pmatrix} \quad A = \begin{pmatrix} 6 & 0 & 5 \\ 0 & 3 & 4 \\ 2 & 1 & 6 \end{pmatrix} \]

\[ 6y_1 = 3y_2 \geq 2y_1 + y_2, \quad y^T = \left( \frac{1}{3}, \frac{2}{3}, 0 \right) \]

\[ x = \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \quad y = \left( \frac{1}{3}, \frac{2}{3}, 0 \right) \text{ is N.E.} \]
Finding mixed N.E.

<table>
<thead>
<tr>
<th>(4,2)</th>
<th>(6,4)</th>
<th>(1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2,1)</td>
<td>(1,2)</td>
<td>(0,1)</td>
</tr>
<tr>
<td>(4,1)</td>
<td>(0,2)</td>
<td>(3,4)</td>
</tr>
</tbody>
</table>

Q: find all N.E. in this game.

Pure N.E.: (row 2, col 2)
(row 3, col 3)

row 1 \leq row 3 \text{ so cannot eliminate}

col 2 \geq col 1 \text{ in } B \text{ so col 1 never used. } \Rightarrow \text{ can eliminate.}

Note: after removing col 1, row 1 \leq \Phi(row 2) + (1-\Phi)(row 3)
with \( t \in (\frac{1}{3}, 1) \)

So can eliminate row 1.
here, no N.E. with one player pure and other mixed. If both mixed, use rows 2, 3 col 2, 3.

\[ x^T = (0, x_2, 1-x_2) \quad y^T = (0, y_2, 1-y_2) \]

\[ y \text{ optimal against } x \Rightarrow \text{ col } 2, 3 \text{ equally good, so } x^TB = (*, v, v) \]

\[ x^TB = (0, x_2, 1-x_2) \begin{pmatrix} 2 & 4 & 1 \\ 1 & 2 & 1 \\ 2 & 4 \end{pmatrix} = (\ldots, 2, x_2 + 4(1-x_2)) \]

\[ 4 - 3x_2 = 2 \quad \boxed{x_2 = \frac{2}{3}} \]

\[ x \text{ opt. against } y \text{ so } Ay = \begin{pmatrix} v' \\ v' \end{pmatrix} \]

\[ \text{note: if we seek a mixed N.E. with these rows + cols but without domination, need to check } x^TB = (v, v, v) \]
\[
\begin{pmatrix}
4 & 0 & 1 \\
2 & 1 & 0 \\
4 & 0 & 3
\end{pmatrix}
\begin{pmatrix}
0 \\
y_2 \\
1-y_2
\end{pmatrix}
=
\begin{pmatrix}
1-y_2 \\
y_2 \\
3-3y_2
\end{pmatrix}
\]

\[y_2 = 3 - 3\sqrt{2} \quad \Rightarrow \quad y_2 = \frac{3\sqrt{2}}{4}\]

\[x^T = (0, \frac{2}{3}, \frac{1}{3}) \quad y^T = (0, \frac{3\sqrt{2}}{4}, \frac{1}{4}) \quad \text{have} \quad x^T B = (1, 2, 2) \quad A y = \begin{pmatrix} \frac{3\sqrt{2}}{4} \\ \frac{3\sqrt{2}}{4} \\ \frac{3}{4} \end{pmatrix}\]

So \((x, y)\) is a mixed N.E.

Safety values!

For P.1, the value of \(A = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \\ 4 & 0 & 3 \end{pmatrix}\) as a 0-sum game.

Value is \(\frac{3\sqrt{2}}{4}\) with safety strat \(x = (0, \frac{3}{4}, \frac{1}{4})\) [not part of a N.E.]

For P.2: Look at \(B^T\) as 0-sum game.

\(B^T = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 1 & 1 & 4 \end{pmatrix}\) Value is 2, safety strat is \((\text{Col 2})\)
If there was no domination, lots of cases to check.

**Thm:** In any game with finite action spaces $E \subseteq \text{N,E}$.  

*First idea:* pick some $(x,y)$

If $x$ not optimal against $y$ replace by $x'$ which is same for $y$.

This gives $(x',y)$. Repeat $(x'',y'')$.

If this stabilizes: done.

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\] never stabilizes.
Key idea: look for a fixed point of some dynamic on strategies.

fixed point of $F: S \to S$ is $u \in S$

st. $F(u) = u.$

Look for $F(x, y) = (x', y')$ st. $x' = x, y' = y$ iff $(x, y)$ is a N.E.
Recall: Brouwer fixed pt theorem!

If $f: K \to K$ is contin. and $K \subset \mathbb{R}^n$ is closed + convex + bounded, then $f$ has a fixed point.

**Spenser's Lemma**

$\delta = \frac{1}{4}$

points on a segment coloured red or blue.

0 is blue, 1 is red

**Lemma:** Some inner interval has endpoints of both colours.

Moreover, the number of such segments is odd.

**Proof:** Since we start with blue and end with red, we switch colours an odd number of times.
triangle is cut into smaller triangles

Each corner has a unique colour
A point on the boundary has a colour of one end point of its edge.

**Lemma**: There is a small triangle with all 3 colours.
Moreover, the number of such triangles is odd.

**Proof**: count \( \triangle \) (triangle, red-blue edge)
A rainbow triangle (all colours) \( \triangle \) has one
A triangle with all red-red-blue or red-blue-blue has 2 R-B edges. All others have none.
Our quantity $Q$ is $Q = n_1 + 2n_2 + 0n_0$.

with $n_i =$ # triangle with $i$ R-B edges

An internal red-blue edge is counted twice.

A boundary R-B edge is counted once.

$Q = \#\{\text{R-B edges on boundary}\} + 2\#\{\text{R-B edges inside}\}$

\underline{odd by 1-dim. lemma.} \hspace{1cm} \underline{even}

So $Q$ is odd.

$n_1 + 2n_2$ is odd so $n_1$ is odd.

$n_i =$ # rainbow triangles.

\hfill \Box
Lemma: The number of rainbow simplices is odd.

proof: count (simplices, R-B-G face)
Brover in $d=2$,

$$k = \Delta$$

$$f: k \to k \text{ cont.}$$

$$f(x, y) = (u, v)$$

If $u < x$ then red

If $u \geq x$ and $v < y$ then blue

otherwise black.

Can find $(z_1, z_2, z_3)$ s.t.

At $z_1$, $x$-coord. decreases

At $z_2$, $y$-coord. decreases

At $z_3$, all coord. get larger,
So can find nearby $z_1, z_2, z_3$.

Make the partition into triangles of size $\varepsilon$. Get

$z_1^{(\varepsilon)} \quad z_2^{(\varepsilon)} \quad z_3^{(\varepsilon)}$

Can assume $z_i^{(\varepsilon)} \xrightarrow{\varepsilon \to 0} z$ (Subsequence Converges)

Can get $z_i^{(\varepsilon)} \rightarrow z$ with

$f(z_i^{(\varepsilon)}) \preceq x$-coord. $\leq$

$x$-coord of $z$.

So for $z$, $x$-coord. is decreasing not increasing.

$z_2^{(\varepsilon)} \rightarrow z$ so $y$-coord at $z$ not increasing.

$z_3^{(\varepsilon)} \rightarrow z$ so both coord do not decrease.
So \( z \) is a fixed point of \( f \).

For general \( K \): place \( K \) inside a \( \triangle \). define \( g : \triangle \to \triangle \)

If \( x \in K \) then \( g(x) = f(x) \).

If \( x \notin K \) then \( g(x) = f(y) \) with \( y \in K \) nearest to \( x \).

\( g \) is cont. so \( g \) has a fixed point.

\( g(x) = x \) so \( x \in K \) so \( f(x) = g(x) = x \).
Monotonicity

In 0-sum games: Increasing $A_{ij}$ can only increase value.

- If $B_{ij} \geq A_{ij}$ then $\text{Val}(B) \geq \text{Val}(A)$
- Adding rows increases Value (greater or equal)
- Adding cols. decreases Value

In general sum games these can fail.

**Chicken**

<table>
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<tr>
<th></th>
<th>Swerve</th>
<th>Drive</th>
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<tbody>
<tr>
<td>Swerve</td>
<td>(-1, -1)</td>
<td>(-2, 1)</td>
</tr>
<tr>
<td>Drive</td>
<td>(1, -2)</td>
<td>(-L, -L)</td>
</tr>
</tbody>
</table>

$L = \text{large}$
Pure N.E.: (D, S) or (S, D)

Mixed N.E.: If P2 uses y, 
\[-y_1 - 2y_2 = 1, y_1 - L' y_2\]
\[2y_1 = (L-2)y_2\]
\[y = \left(\frac{L-2}{L}, \frac{2}{L}\right)^T\]

\[y = x = \text{same is a N.E.}\]

Remove option to swerve for P1:

\[
\Rightarrow \text{Only N.E is (D, S) outcome (1, 2)}
\]

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<td>(-10, 0)</td>
</tr>
<tr>
<td>D</td>
<td>(0, -10)</td>
<td>(-3, -3)</td>
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\[D > C\] for 
both players
Repeated Prisoner's Dilemma (Iterated)

Can play a game in many rounds.
Actions can depend on all choices from previous rounds.

Example strategies: Always C,
Always D.
Fair coin each round.

Tit for Tat: in round 1 C
in round k use opponent’s choice from round k-1

Note: If # rounds T is fixed then in last round
D dominates C.

By induction, optimal strategy is always D (against rational opponent)
Extensive Form of a game (Kuhn tree)

Game states are nodes.
At each node one player decides where to move.
End-points are leaves. Each has an outcome.
E.g. subtraction $\{1, 2\}$

$(1, 1)$: P2 wins

Can have equivalent vertices in many places in the tree
Note: Can assume that players alternate making decisions.

Is equivalent to
Trees, Pirates and Misinformation

Player 1  Player 2

B \Rightarrow (2,4)
E \Rightarrow (1,4)
C \Rightarrow (1,4)
F \Rightarrow (2,3)
D \Rightarrow (2,3)

Player 1 can choose between B or D and get 2.
Player 2 gets 4 or 3.
For any tree can evaluate from bottom up.

Note: Also for DAG (directed acyclic graph)
Jackpot game

Cont: grow pot. Stop: loses gets $\frac{1}{2}$ pot.

By induction from $N$ down to 2,
At every node the better choice is to stop.

Mutually Assured Destruction

Remove option to ignore:
Removing an option improved outcome for p2.

Adding randomness to kuhl trees: some nodes are random, so a child is chosen with given distribution.

**Partial Information**

A player needs to choose action with partial information.

**Information sets**: A player knows which set they are in but not which node inside the set.

E.g.,

```
A   C
A (4,1)
B
```
Fish market

Fishmonger knows if fish is fresh or old.
Buyer does not, but can ask. Seller can lie. Price is 10.

Value for buyer: fresh fish 13
    old fish 0

Seller: +5 if sell fish
    -5 if stuck with old fish

Assume \( \frac{1}{5} \) of the fish are old.
Fish market

\[ \frac{1}{5} \text{ of fish are old.} \]

To buyer, fresh fish worth 12
old worth 0

Price is 10

To seller: stuck with old fish: -5

Seller knows state of fish.
Buyer asks: "is this fresh?"
Seller says Yes/No (can lie)
Buyer Buys or Leaves
Assume: if fresh then seller says so.

Seller strategies:
* Claim fresh always
* honest

Buyer: 4 possible strategies
* always buy
* always leave
* buy only "fresh"
* buy only "old"

Buying a fish known to be old is dominated by leaving.

\( R \) : reputation cost
"Fresh" | Buy | Leave
---|---|---
Honest | $(5, -\frac{2}{5})$ | $(-1, 0)$

Buy: if told fresh

(Fresh, Leave): seller gets $\frac{4}{5} \cdot 0 + \frac{1}{5} (-5) = -1$

(Fresh, Buy): $\frac{4}{5} (5, 2) + \frac{1}{5} (5, -10) = (5, -\frac{2}{5})$

H, B: $\frac{4}{5} (5, 2) + \frac{1}{5} (-5, 0) = (3, \frac{8}{5})$

(Always claim fresh, Always Leave) is a N.E. (only one)
With reputation cost $i$:

| $\left(5 - \frac{8}{5}, -\frac{2}{3}\right)$ | $(-1, 0)$ |
| $\left(3, \frac{8}{5}\right)$ | $(-1, 0)$ |

If $R \geq 10$ then honesty is dominating.

If $R \geq 10$ then (Honest, Buy) is a N.E. as well.
Each player antes 1

P1: Bet 1, more or check
If check: P2 can bet 1 or check
Check: high card wins
bets: P1 call or fold
If bet: P2 can call or fold

After dealing:

depends on which card higher
Full tree:

Dealer

many copies

inf. sets.
Infinite games (all the same)

Setup: A game is repeated \( \infty \) many times.
Your strategy can depend on actions chosen in prev. rounds by everyone.

E.g. 
\[
\begin{array}{cc}
C & P \\
C & (6, 6) & (0, 8) \\
P & (8, 0) & (2, 2) \\
\end{array}
\]

If repeated fixed number of rounds, \( N \), then by induction, only N.E. is always pick P.
Two ways to handle infinity: Get $M_i$ in round $i$

- **Discount:** $M = \sum_{i=0}^{\infty} M_i \beta^i$ for some $\beta \in (0, 1)$

- **Average payoff:**

  $$M = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N} M_i$$

Claim: If $\beta$ small, then should always defect.

In game 0, C costs 2 compared to D.

In games 1, 2, ... best can hope for is 8 each.

Total is $8\beta + 8\beta^2 + 8\beta^3 + \ldots = \frac{8\beta}{1-\beta}$
If $\frac{8\beta}{1-\beta} < 2$ then better to choose D in round 0.

i.e. $\beta < \frac{1}{5}$ \Rightarrow always D is only N.E.

**Claim:** If $\beta \approx 1$ then Tit-for-Tat by both is a N.E.

**Recall:** Tit-for-Tat: C in round 0, then mirror opponent's prev. choice.

Consider any seq. opp. uses TFT

your payoff ($M_i$):

need to show: $\sum_{i=0}^{\infty} M_i \beta^i \leq \sum_{i=0}^{\infty} 6 \beta^i$. 
A block of $P$'s changes your payoff by
\[2\beta^i - 4\beta^{i+1} - 4\beta^{i+2} \ldots - 6\beta^j = \ldots = (\beta^i - \beta^j)(2 - \frac{4\beta}{1-\beta})\]
This is negative if $2 < \frac{4\beta}{1-\beta}$ i.e. $\beta > \frac{1}{3}$. \(\square\)
Claim: In a limit avg payoff model, there is a N.E. with outcomes (7, 3).

<table>
<thead>
<tr>
<th>P1</th>
<th>CDCDCDCDC</th>
<th>plan</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>CCCCCCCC</td>
<td>avg 7</td>
</tr>
<tr>
<td>P1 gets</td>
<td>686868</td>
<td>avg 7</td>
</tr>
<tr>
<td>P2 gets</td>
<td>606060</td>
<td>avg 3</td>
</tr>
</tbody>
</table>

P1 strategy: follow plan. If P2 ever deviates from plan, then pick D for rest of time.

Against this, P2 cannot do better than agree.

P2 strategy: same.
Other sequences give plans with other avg values. There are limits.

If P1 suggests \((CDDDD)\) repeated \((ccc\) P2 get \((6,0,0,0,0)\) repeated, \(\text{avg is } 6/5\)

Any N.E. must give Each player at least their safety value.
Thm: Any pair of values \((u,v)\) can be the outcome of a N.E. if and only if: each is \(\geq\) safety value and \((u,v)\) is inside hull of \(\langle a_{ij}, b_{ij} \rangle\)