Voting (Social Choice)

Several candidates A, B, C...
Voters have preference orders.

1. Voter 1: A > _1 B > _1 C
2: A > _2 C > _2 B
3: C > _3 B > _3 A

Social choice: Given preference orders, pick one candidate.
Social Ranking: " " " rank candidates
Borda count: give i points to ith candidate by each vote
add points.

In e.g. get A: 5  B: 7  C: 6
so A ▷ C ▷ B

A complete solution must decide on an order for every case. (Borda leaves ties unspecified)

Plurality: winner is candidate with most top votes
can get full ranking by removing winner from lists
and repeating proc. for 2nd place, etc.
Setup: Each voter i has a preference order \( \succ_i \) on candidates. Assume this is a complete order. Denote the set of all preferences by \( \Pi = (\succ_1, \succ_2, \ldots, \succ_n) \).

Voting rule is a function \( f(\Pi) \) giving the winner.

Ranking rule: Given \( (\succ_i) = \Pi \) determine a social ranking \( D \).

\( D \): order on candidates.

Examples of voting rules: Winner is candidate who is top choice for most voters [Plurality].
Borda count: Assign weight to candidate in ith pos. add weights from voters.

Runoff: Voters pick 1. If no majority, top 2 candidates run in round 2.

Dictatorship: $f(T_i)$ depends only on the order of one voter $i$.

Note: We consider only non-random methods.

Transferrable votes: Eliminate the candidate with fewest top votes.

Instant runoff: Ballots include ranking of candidates, not just top choice.
Strategic Vulnerability: A voter can get a better outcome by voting other than actual \( v_i \).

Most systems have Strat_Vuln.

Dictatorship does not.

Unanimity: If For ranking \( S.V. \) is if a voter has \( A >_i B \) and \( f(\Pi) \) has \( A > B \) but if voter \( i \) changes their order, get \( A > B \)

Idea: If a group can change outcome, then there is some configuration where a single voter can change D
Unanimity: If all voters have $A >_i B$ then $A > B$

Independence of Irrelevant Alternatives (IIA)

If voters change pos. of candidates other than $A, B$
then combined ranking on $A, B$ unchanged.

$\forall i: A >_i B \iff A >_i B$

then $A > B \iff A > B$

Lemma: If IIA fails then there is strat. vote.

proof: consider two pref. orders $TT, TT'$

Assume in $(TT)$ $A > B$ $f(TT') : B > A$. 

Interpolate, one at a time voters change from \( >_i \) to \( >_i \) at some point switch from \( ADB \) to \( BDA \).

\[ \Rightarrow \exists \text{ some config where a single voter can change } ADB \text{ to } BDA. \]

If \( A >_i B \) also \( A >_i B \) so we found strat. vuln.
IIA: If candidates other than A,B moved in voter rankings, then social order of A,B unchanged:

\[ A \nabla B \iff A \nabla \not{B} \]

Seen: no IIA then strategic voting possible.

Thm (Gibbard-Satterthwaite): The only voting rule that for 3 or more candidates, with no possible strategic voting, and where every candidate can win is the dictatorship.

Dictatorship: There is a voter i s.t. \( f(N) \) is the top candidate in \( \succ_i \).
Note: For 2 candidates there are many systems.

Arrow's theorem: The only ranking rule with IIA and unanimity is the dictatorship.

Idea for proof of G-S: Assume another voting rule with properties exist. Use it to construct a ranking rule that violates Arrow's thm.
* Plurality

* Single Transferable Vote: candidate with fewest votes eliminated, their voters choose to transfer their votes

* Hare quota: use plurality in each riding, party with most seats picks leader.

Q: In STV can we have A>B for all i but B elected?

No.

Strategic voting possible in all 3.

e.g. STV:

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<th>40%</th>
<th>25%</th>
<th>35%</th>
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</thead>
<tbody>
<tr>
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<td>B</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>B</td>
<td></td>
</tr>
</tbody>
</table>

C wins
If some change $A \to B \to C$

<table>
<thead>
<tr>
<th>11</th>
<th>29</th>
<th>25</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>A</td>
<td>B</td>
<td>C</td>
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<tr>
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<td>B</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>C</td>
<td>A</td>
<td>B</td>
</tr>
</tbody>
</table>

Thm: $\#$ Majority is Stablest

(Least likely to change outcome as result of noise in sampling)

Noise: errors, voter turnout...