Problem 1. Consider a game played on the following graph. An arrow from $x$ to $y$ means it is allowed to move from $y$ to $x$. (The same graph is shown twice.)

(a) Find the P and N positions.
(b) If the game starts at position $a$, who wins (first or second player)?
(c) Find the Sprague-Grundy value of each state.

Solution.  (a,c) These are done from the bottom up. See image.

(b) This is an N-position, so first player wins by moving to $e$.

Problem 2. A chess queen is placed on a chess board, but is allowed to move only left, down, or diagonally left-down. (There is no move from the bottom left corner.) Find the Sprague-Grundy value of each position in a $5 \times 5$ board.

Solution.  These are computed from the bottom up. The result is shown below.

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Problem 3. Consider the subtraction game with subtraction set \{2, 3, 4, 5\}.
(a) Find the Sprague-Grundy value for piles of size up to 13.
(b) If there are piles of sizes \{6, 7, 8, 9\}, (and we are allowed to remove 2,3,4, or 5 chips from one of the piles), find a winning move.
(c) For the same position, find all winning moves.
(d) What is \(g(344)\)?

Solution. (a) Starting from the terminal states \(\{0,1\}\) we find the sequence of values:

| \(n\) | 0 1 2 3 4 5 6 7 8 9 10 11 12 13 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \(g(n)\) | 0 0 1 1 2 2 3 0 0 1 1 2 2 3 |

(b,c) We have 
\[g(6) \oplus g(7) \oplus g(8) \oplus g(9) = 3 \oplus 0 \oplus 0 \oplus 1 = 2.\] 
The possible winning moves are 6 \rightarrow 2, 6 \rightarrow 3, 7 \rightarrow 4, 7 \rightarrow 5, 8 \rightarrow 4, 8 \rightarrow 5, 9 \rightarrow 6.
(d) \(g(n)\) has period 7, observed above and continues since the maximal move is 6 and we observe the period for more than 6 steps. So \(g(344) = g(1) = 0.\)

Problem 4. (a) Find all optimal strategies for each player in the zero-sum game with matrix \(A\) below.
(b) Find some optimal strategy for each player in the zero-sum game with matrix \(B\) below.

\[
A = \begin{pmatrix} 0 & 2 & 3 & 5 & 8 \\ 8 & 5 & 3 & 2 & 0 \end{pmatrix} \quad \quad B = \begin{pmatrix} 6 & 5 & 2 & 1 \\ 5 & 4 & 3 & 0 \\ 1 & 0 & 5 & 4 \\ 2 & 0 & 2 & 3 \end{pmatrix}
\]

Solution. (a) If Player 1 uses strategy \((x, 1−x)\) then the value she always gets is
\[
\min(8(1−x), 2x + 5(1-x), 3, 5x + 2(1-x), 8x) = \min(8 - 8x, 5 - 3x, 3, 2 + 3x, 8x).
\]
This is always at most 3, and is equal to 3 if all of 8 - 8x, 5 - 3x, 2 + 3x, 8x are at least 3. This happens for \(\frac{3}{8} \leq x \leq \frac{5}{8}\), so the optimal strategies for Player 1 are \((x, 1−x)\) with that range of \(x\). The only optimal strategy for Player 2 is \((0, 0, 1, 0, 0)\). For example, if Player 1 chooses \((1/2, 1/2)\), his payment vector is \((4, 7/2, 3, 7/2, 4)\), so he loses unless he picks the middle column.
(b) Column 1 is dominated by column 2, so remove it. Then dominate row 4 by row 3, column 3 by column 4, and row 2 by row 1, leaving \(\begin{pmatrix} 5 & 1 \\ 0 & 4 \end{pmatrix}\). By the principle of invariance, \((1/2, 1/2)\) gives Player 1 a value \(V = 5/2\). Player 2 achieves 5/2 by \((3/8, 5/8)\). In the original game these are \((1/2, 0, 1/2, 0)\) and \((0, 3/8, 0, 5/8)\).

Problem 5. Find the value and an optimal strategy for the game \(A = \begin{pmatrix} 0 & 6 \\ 4 & 6 \end{pmatrix}\) for an arbitrary value \(s\). Draw a graph of the value as a function of \(s\) (from \(-\infty\) to \(\infty\)).

Solution. If \(s \geq 4\) then 4 is a saddle point and the value is 4. If \(s \leq 4\), then the principle of invariance can be used. If Ruth uses \((p, 1−p)\) then Chris chooses from \((4(1−p), 6p + s(1−p))\). These are equal if \(p = \frac{4−s}{10−s}\) and the value is \(\frac{24}{10−s}\).
Problem 6. Find a hyperplane which separates 0 from the set $K \in \mathbb{R}^3$ given by $K = \{xyz \geq 1, x, y, z \geq 0\}$.

Solution. There are many possibilities. One is $x + y + z = 1$, since if $xyz \geq 1$ then at least one of them is 1 and the others are positive. Thus for $(x, y, z) \in K$ we have $x + y + z > 1$. 
