Problem 1. A σ-algebra $\mathcal{F}$ is said to be generated by a partition if there is some partition $\mathcal{B} = \{B_i\}$ of Ω so that every set $A \in \mathcal{F}$ is a union of some parts in the partition, and every such union is in $\mathcal{F}$.

(a) If $\mathcal{A} \subset 2^\Omega$ is finite, show that the generated σ-algebra has $|\sigma(\mathcal{A})| \le 2^{2^{|\mathcal{A}|}}$.

(b) Show that any σ-algebra on a countable set Ω is generated by a partition of Ω.

Problem 2. Give an example of a measure space $(\Omega, \mathcal{F})$ and function μ on $\mathcal{F}$ that is additive but not σ-additive, i.e. $\mu(\cup A_i) = \sum \mu(A_i)$ for a finite collection of disjoint A_i, but not for some infinite collections.

Problem 3. What is the σ-algebra generated by all singletons $\{x\}$ for $\Omega = \mathbb{R}$?

Problem 4. Show that the following collections generate the same σ-algebra (Borel) on $\mathbb{R}$:

- Open intervals: $\{(a, b) : a < b\}$.
- Closed intervals: $\{[a, b] : a < b\}$.
- Half open intervals: $\{(a, b] : a < b\}$.
- Half-lines: $\{[a, \infty) : a \in \mathbb{R}\}$.

Problem 5. For a function $f : [0, 1] \to \mathbb{R}$, let C be the set of points where f is continuous. Prove that C is in the Borel σ-algebra.

Problem 6. A permutation σ is called a **derangement** if $\forall i, \sigma(i) \neq i$. Consider a uniform random permutation σ of $\{1, \dots, n\}$, and let D_n be the event that σ is a derangement. Use the inclusion-exclusion principle to find a formula for the number of derangements, and show that $\mathbb{P}(D_n) \xrightarrow[n\to\infty]{} e^{-1}$.

Problem 7. Consider the space $\Omega = \{0, 1\}^{\mathbb{N}}$ of binary sequences (ω_i), with the product probability measure $\mathbb{P}$ where $\mathbb{P}(\omega_i = 1) = 1/2$. Let R_n be the longest consecutive run of 1s in the first n terms. For example, if $\omega = (1, 0, 1, 1, 1, 0, 1, 1, \dots)$ then $R_4 = 2$ and $R_8 = 3$.

Prove that almost surely $\lim_{n\to\infty} \frac{R_n}{\log_2 n} = 1$.