Problem 1. A σ-algebra $\mathcal{F}$ is said to be generated by a partition if there is some partition $\mathcal{B} = \{B_i\}$ of Ω so that every set $A \in \mathcal{F}$ is a union of some parts in the partition, and every such union is in $\mathcal{F}$.

(a) If $\mathcal{A} \subset 2^\Omega$ is finite, show that the generated σ-algebra has $|\sigma(\mathcal{A})| \leq 2^{2^{|\mathcal{A}|}}$.

(b) Show that any σ-algebra on a countable set Ω is generated by a partition of Ω.

Solution.

(a) Let $\mathcal{A} = \{A_1, \dots, A_n\}$ and let each B_i be either A_i or A_i^c. There are 2^n choices for the B_i's. For each such sequence of Bs, let S be their intersection, so there are 2^n possible sets S. (Some of them may be empty.) Moreover, the sets S are a partition of Ω. There are 2^{2^n} sets that can be written as unions of sets S_i, and these are a σ-algebra containing each A_i.

(b) For a countable set Ω, define an equivalence relation $x \sim y$ if there is no set including one but not the other. This is an equivalence relation, so the equivalence classes are a partition of Ω. Moreover, there are at most countably many equivalence classes. We need to show that each equivalence class S is in the σ-algebra. To see this, for each $x \in \Omega \setminus S$ take a set A_x such that $S \subset A_x$ but $x \notin A_x$. Such a set exists since otherwise x is in the equivalence class. The countable intersection of the sets A_x is S.

Problem 2. Give an example of a measure space $(\Omega, \mathcal{F})$ and function μ on $\mathcal{F}$ that is additive but not σ-additive, i.e. $\mu(\cup A_i) = \sum \mu(A_i)$ for a finite collection of disjoint A_i, but not for some infinite collections.

Solution. There are many constructions based on the axiom of choice. One is to take a non-principle ultra-filter on $\mathbb{N}$: a collection of subsets closed to finite intersection, that contains exactly one of A, A^c and not any finite set. Then let $\mathbb{P}(A) = 1$ if A is in the ultra-filter.

Another is bsaed on the Banach limit: a linear extension of lim to all bounded sequences. For a set in $\mathbb{N}$, take $\mathbb{P}(A) = \lim 1_{n \in A}$. For any finite set this is 0, but for the countable union $\mathbb{N}$ it is 1.

A similar example on $\mathbb{R}$ is $\lim n^{-1} \mu(A \cap [0, n])$, where μ is Lebesgue measure.

Problem 3. What is the σ-algebra generated by all singletons $\{x\}$ for $\Omega = \mathbb{R}$?

Solution. Any countable set must be in $\mathcal{F}$. Therefore let $\mathcal{F}$ be the σ-algebra of sets that are countable or co-countable (with A^c countable). This is easily a σ-algebra which contains all singletons, so it is the solution.

Problem 4. Show that the following collections generate the same σ-algebra (Borel) on $\mathbb{R}$:

- Open intervals: $\{(a, b) : a < b\}$.
- Closed intervals: $\{[a, b] : a < b\}$.
- Half open intervals: $\{(a, b] : a < b\}$.
- Half-lines: $\{[a, \infty) : a \in \mathbb{R}\}$.

Solution. Let $\mathcal{F}_i$ for $i = a, b, c, d$ be the resulting σ algebra. The closed interval is $[a, b] = \cap_n (a - 1/n, b + 1/n)$, so $[a, b] \in \mathcal{F}_a$. Therefore $\mathcal{F}_b \subset \mathcal{F}_a$.

Similarly, $(a, b] = \cup [a + 1/n, b]$ so $(a, b] \in \mathcal{F}_b$ and so $\mathcal{F}_c \subset \mathcal{F}_b$.

The other inclusions are similar. We can write $[a, \infty)$ as a union of closed intervals. To go from half lines to finite intervals, take the difference $[a, \infty) \setminus [b, \infty) = [a, b)$.

Problem 5. For a function $f : [0, 1] \to \mathbb{R}$, let C be the set of points where f is continuous. Prove that C is in the Borel σ-algebra.

Solution. Let $A_{i,n,m}$ be the event that on the interval $I_{i,n} = (\frac{i}{n}, \frac{i+1}{n})$, the minimal and maximal values of f differ by at most $1/m$.

The function f is continuous at $x \in (0,1)$, if and only if for every m there is some n and i so that $x \in I_{i,n}$ and $A_{i,n}$ holds. Therefore

$$C \cap (0,1) = \bigcap_m \bigcup \{I_{i,n} \text{ s.t. } A_{i,n,m}\}.$$

This is an intersection of unions of open intervals, so is a Borel set.

Adding 0 or 1 if f is continuous there keeps the set C measurable.

Problem 6. A permutation σ is called a **derangement** if $\forall i, \sigma(i) \neq i$. Consider a uniform random permutation σ of $\{1, \dots, n\}$, and let D_n be the event that σ is a derangement. Use the inclusion-exclusion principle to find a formula for the number of derangements, and show that $\mathbb{P}(D_n) \xrightarrow[n\to\infty]{} e^{-1}$.

Solution. Let A_i be the event that $\sigma(i) = i$, so that $\mathbb{P}(A_i) = 1/n$. For any set of k indices, we have

$$\mathbb{P}(A_{i1} \cap \dots \cap A_{i_k}) = \frac{(n-k)!}{n!},$$

since there are $(n-k)!$ permutations keeing these k indices fixed. There are $\binom{n}{k}$ possible such sets. By inclusion-exclusion we have

$$\mathbb{P}(\cup_i A_i) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{1!} = \frac{1}{-} \frac{1}{2!} + \frac{1}{3!} \cdots \pm \frac{1}{n!}.$$

This converges to $1 - 1/e$.

Problem 7. Consider the space $\Omega = \{0,1\}^{\mathbb{N}}$ of binary sequences (ω_i), with the product probability measure $\mathbb{P}$ where $\mathbb{P}(\omega_i = 1) = 1/2$. Let R_n be the longest consecutive run of 1s in the first n terms. For example, if $\omega = (1,0,1,1,1,0,1,1,\dots)$ then $R_4 = 2$ and $R_8 = 3$.

Prove that almost surely $\lim_{n\to\infty} \frac{R_n}{\log_2 n} = 1$.

Solution. Fix some $\varepsilon > 0$. We first bound the probability that $R_n \geq k = [(1+\varepsilon)\log_2 n]$. There are n positions where a long run of 1s may start. (Actually $n - k + 1$, which is slightly less.) Each is the start of a k-run with probability 2^{-k}. We have $2^{-k} \leq 2n^{-(1+\varepsilon)}$ (the factor of 2 comes from the rounding down). By the union bound,

$$\mathbb{P}(R_n \geq k) \leq n2^{-k} \leq n \cdot 2n^{-(1+\varepsilon)} = 2n^{-\varepsilon}.$$

If $\varepsilon > 1$ then Borel-cantelly shows that $R_n > k$ only happens finitely many times. To get this for any $\varepsilon > 0$, consider first just the sequence $n_i = [(1+\delta)^i]$. Along this subsequence, $\mathbb{P}(R_n \geq k)$ is summable, so eventually $R_n \leq k$ for $n = [(1+\delta)^i]$. To get the remaining n's we use monotonicity of R_n in n. For $n \in [n_i, n_{i+1}]$ we have $R_n \leq R_{n_{i+1}}$, which implies

$$R_n \leq (1+\varepsilon)\log_2 n_{i+1} \leq (1+\varepsilon)[\log_2(1+\delta) + \log_2 n].$$

By taking δ small, this is less than $(1+2\varepsilon)\log_2 n$.

Therefore eventually $R_n \leq (1+2\varepsilon)\log_2 n$. Since ε was arbitrary, this gives the upper bound.

The lower bound is easier: Let $\varepsilon > 0$ and $k = (1-\varepsilon)\log_2 n$. Split $1 \dots n$ into n/k segments of length k. Each is all ones with probability 2^{-k} independently, so the probability that none of them are all ones is $(1-2^{-k})^{n/k}$. Therefore

$$\mathbb{P}(R_n \leq k) \leq (1-2^{-k})^{n/k} \leq (1-n^{\varepsilon-1})^{n/k}.$$

This is summable, so by Borel-Cantelli eventually $R_n \geq (1-\varepsilon)\log_2 n$.