Problem 1. (a) Find n events so that any $n-1$ are independent but all together are not.
(b) Show that if these events are in some probability space Ω then Ω has at least 2^{n-1} points.

Problem 2. (a) Let $(A_i)_{i\leq n}$ be independent events with probability $1/n$ each. Find the probability that exactly k of them occur, and the limit as $n\to\infty$.
(b) Consider a random uniform permutation $\sigma\in S_n$. What is the probability that it has exactly k fixed points? What is the limit as $n\to\infty$?
(c) Is there a direct connection between parts a and b?

Problem 3. Let A_n be independent events with $\mathbb{P}(A_n)\neq 1$ for every n. Prove that $\mathbb{P}(\cup A_n)=1$ if and only if $\mathbb{P}(A_n \text{ i.o.})=1$.

Definition 1. The m-ary tree is a graph with a single vertex on level 0. Each vertex on level k has m edges to vertices on level $k+1$, all distinct (so there are m^k vertices on level k).

Problem 4. Consider percolation on the m-ary tree, where each edge is open independently with probability p. Let C_0 be the cluster of the root vertex. Prove that if $p<1/m$ then $\mathbb{P}(|C_0|=\infty)=0$.
(*) Prove the same also for $p=1/m$.

***Problem 5.** Construct a graph with $0<p_c<1$ where $\theta(p_c)>0$. Prove your claims. (There are many different solutions.)

Problem 6. If X_n are independent random variables on some probability space, show that $\mathbb{P}(\sum X_n \text{ converges})\in\{0,1\}$.

Problem 7. Let $X_1,X_2,\dots$ be i.i.d. (independent and identically distributed) random variables with standard exponential distribution:

$$\mathbb{P}(X_i>t)=e^{-t} \qquad \text{for } t>0.$$

For any $a>0$ compute $\mathbb{P}(X_n>a\log n \text{ i.o.})$. Deduce that $\limsup \frac{X_n}{\log n}=1$ a.s.

Problem 8. Show that for any sequence of random variables X_n there is a sequence of constants $a_n>0$ so that $a_nX_n\to 0$ a.s.