
Problem 1. (a) Find n events so that any $n-1$ are independent but all together are not.
(b) Show that if these events are in some probability space Ω then Ω has at least 2^{n-1} points.

Solution.
(a) Consider $n-1$ independent fair coins. Let $n-1$ of the events be that coin i is heads. Let the last event be that the total number of heads is odd. It is easy to verify that any $n-1$ are independent, but all are not.
This is equivalent to the probability space where $\mathbb{P}$ is the uniform distribution on the 2^{n-1} vectors in $\{0,1\}^n$ with even sum, and the indicator of A_i is coordinate i.
(b) Intersections of A_i or A_i^c for $i = 1,\dots,n-1$ gives 2^{n-1} disjoint events. If none of the events has probability $\{0,1\}$, then these events have non-zero probability, and so $|\Omega| \geq 2^{n-1}$.
If some A_i has probability 0 or 1, then the other $n-1$ events are independent, but then A_i is also independent of all of them. (Exercise: a trivial event is independent of everything), so all n events are independent, which we know is not the case.

Problem 2. (a) Let $(A_i)_{i\leq n}$ be independent events with probability $1/n$ each. Find the probability that exactly k of them occur, and the limit as $n \to \infty$.
(b) Consider a random uniform permutation $\sigma \in S_n$. What is the probability that it has exactly k fixed points? What is the limit as $n \to \infty$?
(c) Is there a direct connection between parts a and b?

(a) The probability is $\binom{n}{k}(1/n)^k(1-1/n)^{n-k}$. For k fixed, as $n \to \infty$ this tends to $e^{-1}/k!$.
(b) By inclusion exclusion, the probability a permutation has no fixed point is $p_n = (1 - 1 + 1/2! - 1/3! + \cdots \pm 1/n!) \sim 1/e$. The probability that k specific points are the fixed points is $\frac{(n-k)!}{n!}p_{n-k}$. Multiplying by $\binom{n}{k}$ possibilities gives $\frac{p_{n-k}}{k!}$, which also tends to $e^{-1}/k!$.
(c) In both cases there are independent or almost independent events with total probability 1. The number of such events that occur converges in many such cases to Poisson, and is part of a general theme of Poisson approximations. This is an analogue of the CLT for rare events.

Problem 3. Let A_n be independent events with $\mathbb{P}(A_n) \neq 1$ for every n. Prove that $\mathbb{P}(\cup A_n) = 1$ if and only if $\mathbb{P}(A_n \text{ i.o.}) = 1$.

Solution. Clearly A_n i.o. implies that some of the events occur.

To prove the converse, assume $\mathbb{P}(A_n \text{ i.o.}) = 0$ (it must be 0 or 1 as a tail event). By Borel-Cantelli, this implies $\sum \mathbb{P}(A_n) < \infty$. We can find some N so that $\sum_{n>N} \mathbb{P}(A_n) < 1/2$. By independence,

$$\mathbb{P}\left(\cap A_n^c\right) = \prod_{n\leq N} \mathbb{P}(A_n^c) \cdot \mathbb{P}\left(\cap_{n>N} A_n^c\right) > 0.$$

The product is positive since $\mathbb{P}(A_n) < 1$. The last term is at least $1/2$ by a union bound:

$$\mathbb{P}\left(\cup_{n>N} A_n\right) \leq \sum_{n>N} \mathbb{P}(A_n) < 1/2.$$

Definition 1. The m-ary tree is a graph with a single vertex on level 0. Each vertex on level k has m edges to vertices on level $k+1$, all distinct (so there are m^k vertices on level k).

Problem 4. Consider percolation on the m-ary tree, where each edge is open independently with probability p. Let C_0 be the cluster of the root vertex. Prove that if $p < 1/m$ then $\mathbb{P}(|C_0| = \infty) = 0$.
(*) Prove the same also for $p = 1/m$.

Solution. Each vertex on level n is connected to 0 with probability p^n. Thus the expected number of such vertices connected is $(pm)^n$. If $p < 1/m$ then by a union bound, $\mathbb{P}(0 \to \infty) < (pm)^n$ for every n so must be 0.

For the bonus: Let $\theta = \theta(p) = \mathbb{P}(0 \to \infty)$. Then $(1-\theta) = (1-p\theta)^m$, since the probability of being connected through each child is $p\theta$ and these are independent. If $p \leq 1/m$ the only solution to this in $[0,1]$ is $\theta = 0$, since $(1-x)^m \geq 1 - mx$ with equality only for $x = 0$.

***Problem 5.** Construct a graph with $0 < p_c < 1$ where $\theta(p_c) > 0$. Prove your claims. (There are many different solutions.)

Solution. See some problems in the next assignment.

Problem 6. If X_n are independent random variables on some probability space, show that $\mathbb{P}(\sum X_n \text{ converges}) \in \{0,1\}$.

Solution. Convergence does not depend on the first n variables, so is a tail event. The claim follows from the Kolmogorov 0-1 law.

Problem 7. Let $X_1, X_2, \dots$ be i.i.d. (independent and identically distributed) random variables with standard exponential distribution:

$$\mathbb{P}(X_i > t) = e^{-t} \qquad \text{for } t > 0.$$

For any $a > 0$ compute $\mathbb{P}(X_n > a \log n \text{ i.o.})$. Deduce that $\limsup \frac{X_n}{\log n} = 1$ a.s.

Solution. Fix $a > 1$, and note that $\mathbb{P}(X_n > a \log n) = e^{-a \log n}$ is summable. Therefore eventually $X_n \leq a \log n$. This implies $\limsup \frac{X_n}{\log n} \leq a$ a.s., and since $a > 1$ was arbitrary, the $\limsup$ is at most 1.

For the lower bound, fix $a < 1$, and note that $\mathbb{P}(X_n > a \log n) = e^{-a \log n}$ is not summable. Therefore $X_n \geq a \log n$ infinitely often. This implies $\limsup \frac{X_n}{\log n} \geq a$ a.s., and since $a < 1$ was arbitrary, the $\limsup$ is exactly 1.

Problem 8. Show that for any sequence of random variables X_n there is a sequence of constants $a_n > 0$ so that $a_n X_n \to 0$ a.s.

Solution. For a random variable X and any $\varepsilon > 0$ we can find M so that $\mathbb{P}(X \geq M) \leq \varepsilon$. This is since $\mathbb{P}(X \leq t) = F(t) \to 1$ as $t \to \infty$.

For X_n, take some M_n so that $\mathbb{P}(X_n \leq M_n) \leq e^{-n}$ (or any summable sequence of probabilities). By Borel-Cantelli, a.s. eventually $X_n \leq M_n$, and therefore $X_n/(nM_n) \to 0$ a.s.