Problem 1. Consider percolation on a graph with vertex set $\mathbb{N}$ with m_n parallel edges between n and $n+1$.

(a) Determine a necessary and sufficient condition on the sequence (m_n) for which there is an infinite connected cluster.

(b) Use this to find a sequence so that $p_c = 1/2$ and at $1/2$ there is a.s. an infinite cluster.

Problem 2. Let $X_1, X_2, \dots$ be i.i.d. random variables with standard normal distribution, and $M_n = \max(X_1, \dots, X_n)$.

(a) Find $\limsup \frac{X_n}{\sqrt{\log n}}$.

(b) Show that $M_n/\sqrt{\log n}$ converges a.s. to the same value. (Hint: You need some estimates on the tail of the normal distribution.)

Problem 3. Show that for any random variable with finite mean, $t\mathbb{P}(X > t) \to 0$ as $t \to \infty$. Moreover, if $\mathbb{E}|X|^a < \infty$ then $t^a\mathbb{P}(X > t) \to 0$.

Problem 4. Let X_i be i.i.d. random variables, and $S_n = \sum_{i \le n} X_i$.

(a) Show that if $\mathbb{E}X_i = +\infty$ then $\frac{1}{n}S_n \to \infty$ a.s.

(b) Show that if $\mathbb{E}X_i$ is not defined (positive and negative parts both infinite) then a.s. $\limsup \frac{1}{n}S_n = \infty$ and $\liminf \frac{1}{n}S_n = -\infty$.

Definition 1. A random variable has density f if it's distribution function is $F(t) = \int_{-\infty}^t f(x)dx$. (In that case, $f(x) = F'(x)$.) A random variable with a density is called continuous.

Problem 5. If X is a continuous random variable with distribution function F and density $f = F'$, show (from the definition of expectation in class) that $\mathbb{E}X = \int_{-\infty} xf(x)dx$.

Definition 2. A pair of random variables X, Y has join density function $f : \mathbb{R}^2 \to \mathbb{R}$ if $\mathbb{P}((X,Y) \in A) = \iint_A f(x,y)\, dx\, dy$.

Problem 6. Suppose continuous random variables X_1, X_2 have densities f_1, f_2 respectively. Prove from the definition of independence that they are independent if and only if they have a joint density of the form $f(x,y) = f_1(x)f_2(y)$.

Problem 7. Let X, Y are two independent random variables. For an angle θ, the rotation by theta is the pair of random variables

$$X' = X\cos\theta + Y\sin\theta, \qquad Y' = -X\sin\theta + Y\cos\theta.$$

Find a joint distribution of independent variables X, Y, with $\mathbb{P}(X = 0) < 1$ so that for any θ the rotation (X', Y') has the same (joint) distribution as (X, Y).
Bonus: Find (with proof) all such distributions.

***Problem 8.** (a) For given $p \in (0,1)$, construct a tree where $p_c = p$.

(b) Find a tree where $p_c = p$ and $\theta(p_c) > 0$.

(Hint: if a vertex in level n has d_n children, write recursions for the probability that 0 is connected to level n, and analyse these.