Problem 1. The Total variation distance $d_{TV}(X,Y)$ is defined as $\inf \mathbb{P}(X' \neq Y')$ over all couplings of X, Y. How does convergence in the total variation distance relate to the other notions of convergence discussed in class? (Including distribution, probability, a.s., L^1.)

Solution. Convergence is total variation implies convergence in probability and therefore also in distribution since $\mathbb{P}(|X-Y| \geq \varepsilon) \leq \mathbb{P}(X \neq Y)$.

It does not imply convergence a.s.: Let X_n be independent Bernoulli$(1/n)$ then they do not converge a.s. but converge in T.V. In fact, $a_n X_n$ converges to 0 in total variation and a_n can be large, so $\mathbb{E}|X_n|^p$ can tend to ∞.

Problem 2. Let X_n be i.i.d. uniform in $[0,1]$ random variables. Let $Y_n = 1$ if X_n is a local maximum: $X_n = \max\{X_n, X_{n-1}, X_{n+1}$. Let $S_n = \sum_{i=1}^n Y_i$. Find constants a, b so that the sum of Y_n has a CLT:

$$\frac{S_n - an}{b\sqrt{n}} \to N(0,1),$$

and prove this convergence. (Hint: The Y_n are not independent, but only depndent for nearby n's. Approximate S_n by a sum of independent R.V.s.)

Solution. There are several approaches. The key is to split S_n into sums of inependent things, but it helps to allow a small error.

First we find $\mathbb{E}S_n \sim n/3$ since $\mathbb{E}Y_i = 1/3$. Next, the second moment: $\operatorname{Var} S_n = \sum_{i,j} \operatorname{Cov}(Y_i, Y_j)$. This sum has n terms with $i = j$ where $\operatorname{Cov}(Y_i, Y_i) = 2/9$. There are about $2n$ terms with $j = i \pm 1$ for which $\operatorname{Cov} Y_i, Y_j) = -1/9$. There are about $2n$ terms with $j = i \pm 2$ for which $\operatorname{Cov} Y_i, Y_j) = 2/15$. (Find this by checking the 5! orders of 5 consecutive elements.) Finally, the rest are independent so 0 covariance. Combined we get $\operatorname{Var}(S_n) = 4n/15 + O(1)$.

Therefore we aim to show $\frac{S_n - n/3}{\sqrt{4n/15}} \to N(0,1)$. One way to do this is to split $[1, n]$ into intervals $I_1 \dots, I_k$ of length $o(n)$ with a gap of two indices between intervals. Say intervals of length $n/\log n$ with $k = \log n$. Let A_i be the sum over I_i. Then $\mathbb{E}A_i = |I_i|/3$ and $\operatorname{Var}(A_i) = 4/15|I_i|$. You can use the triangular array CLT (Lindeberg-Feller) to get that $\sum A_i$ is roughly normal. Moreover, the difference $S_n - \sum A_i$ is at most 2 for each segment so is $o(\sqrt{n})$ and neglegible.

Problem 3. Prove that φ_X is periodic if and only if X takes values in $a\mathbb{Z}$ (multiples of a) for some a.

Solution. If $X \in a\mathbb{Z}$ then $\varphi(t + 2\pi/a) = \varphi(t)$ since $(2\pi/a)X$ is a multiple of 2π.

Conversely, if φ has period b, then $\varphi(b) = \varphi(0) = 1$, so $\mathbb{E}e^{ibX} = 1$. Since $|e^{ibX}| \leq 1$ this implies $e^{ibX} = 1$ a.s., which implies X is a multiple of $2\pi/b$.

Problem 4. Write a proof of the following statement from class: If $X \geq 0$ is a R.V. then $\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt$. Moreover $\mathbb{E}X^a = \int_0^\infty t^{a-1}\mathbb{P}(X > t)dt$.

Solution. For a simple random variable this is verified by rearranging the sums. If $X = \sum a_i 1_{A_i}$ then let $B_i = \bigcup_{j>i} A_j = \{X > a_i\}$. Then $\mathbb{P}(X > t) = \mathbb{P}(B_i)$ on $[a_i, a_{i+1}]$, and continue from that.

For a general random variable, let X_n be simple increasing tending to X, then $\mathbb{P}(X_n > t) \to \mathbb{P}(X > t)$ for almost all t, and so the integrals converge.

For the moments, apply the above to $Y = X^a$: $\mathbb{E}X^a = \int_0^\infty \mathbb{P}(X^a > t)dt$. Then use a change of variable $t = s^a$.

Problem 5. Prove that for any random variable with finite mean, $t\mathbb{P}(X > t) \to 0$ as $t \to \infty$. Moreover, if $\mathbb{E}|X|^a < \infty$ then $t^a\mathbb{P}(X > t) \to 0$.

Solution. For the first claim, let $X_t = t1_{X>t}$, then $X_t \leq X$ and $X_t \to 0$ almost surely. By dominated convergence, $t\mathbb{P}(X > t) = \mathbb{E}X_t \to 0$.

The second part is the same with $X_t = t^a 1_{X>t}$.

Problem 6. If $X_n \xrightarrow{prob} X$ are independent, show that X must be a constant random variable.

Solution. Suppose for a contradiction that X is not constant, so that there are some $a < b$ and ε so that $\mathbb{P}(X < a) \geq \varepsilon$ and $\mathbb{P}(X > b) \geq \varepsilon$. Let $\delta = (b-a)/3$. Convergence in probability implies convergence in distribution, so eventually $\mathbb{P}(X_n < a) \geq \varepsilon/2$ and the same for $\mathbb{P}(X_n > b)$. Therefore with probability $(\varepsilon/2)^2$, for m, n large we have that $X_n < a$ and $X_m > b$ With probability tending to 1, $|X_n - X| \leq \delta$, and same for X_m. This means there is positive probability that all of these occur, which is impossible.

It is also possible to use that there is a subsequence that convereges almost surely, but infinitely often we have $X_n < a$ and also infinitely often $X_n > b$.

Problem 7. Show that convergence in probability is equivalent to convergence in the metric

$$d(X,Y) = \mathbb{E}(|X-Y| \wedge 1).$$

Solution. Note that $\mathbb{E}|X-Y| \wedge 1 \geq \varepsilon \mathbb{P}(|X-Y| \geq \varepsilon)$. If $d(X_n, X) \to 0$, then $\mathbb{P}(|X_n - X| > \varepsilon) \leq d(X_n, X)/\varepsilon$ tends to 0.

Conversely, if $X_n \to X$ in probability, then $|X_n - X| \wedge 1 \to 0$ in distribution. Dominated convergence implies $\mathbb{E}|X_n - X| \wedge 1 \to 0$.