Problem 1. Find a random variable X, and two σ-algebras $\mathcal{G}, \mathcal{H}$ so that $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) \neq \mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G})$. (Hint: a finite Ω is possible).

Problem 2. If a random variable has characteristic function $\varphi(t) = (1 - |t|)^+$, find its density.

Problem 3. If X_n converge in distribution to X, and Y_n converge in probability to 0, show that $X_n + Y_n$ converge in distribution to X.

Problem 4.

- If $X_n \to X$ and $Y_n \to Y$ in probability show that $X_n + Y_n \to X + Y$ in probability.
- Do the same for convergence in L^p for $p \geq 1$.
- Show this fails for convergence in L^p for $p < 1$.
- Show this fails for convergence in distribution.

Problem 5. Suppose random variables (X, Y) have joint density $f(x, y)$ on $\mathbb{R}^2$. Let $Z(y) = \int_{\mathbb{R}} x f(x, y) dx / \int_{\mathbb{R}} f(x, y) dx$. Show that $Z = \mathbb{E}(X|Y)$ using the definition.

Problem 6. Construct a martingale X_n so that $X_n \to \infty$ a.s.

Problem 7. Do not hand this in; a related problem will be in the exam instead. Let X_i be i.i.d. bounded integer random variables, and let $S_n = \sum_{i \leq n} X_i$. Suppose $\mathbb{P}(X > 0)$ and $\mathbb{P}(X < 0)$ are both non-zero, and $\mathbb{E}X \neq 0$.

(a) Show that there is some $0 < q \neq 1$ so that q^{S_n} is a martingale.
(b) Find that q when X_n is ± 1 with probabilities $p, 1 - p$.
(c) Use this to find $\mathbb{P}_k(T_n < T_0)$, where $T_a = \inf\{t : X_t = a\}$.

****Problem 8.** If X_n are independent random variables such that $\sum \mathbb{E}X_n$ exists, and that $\sum X_n$ converges a.s., must it be that $\mathbb{E} \sum X_n = \sum \mathbb{E}X_n$?