Problem 1. Find a random variable X, and two σ-algebras $\mathcal{G}, \mathcal{H}$ so that $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) \neq \mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G})$. (Hint: a finite Ω is possible).

Solution. Anything works. Minimal example: $\Omega = \{0, 1, 2\}$, with $\mathcal{G} = \sigma(\{0\})$ and $\mathcal{H} = \sigma(\{2\})$. Take $\mathbb{P}$ uniform on Ω and $X(\omega) = \omega$.

Writing (a, b, c) for a function with value a, b, c at $0, 1, 2$, we have

$$\mathbb{E}(X|\mathcal{G}) = (0, 3/2, 3/2) \qquad \mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = (3/4, 3/4, 3/2),$$

and

$$\mathbb{E}(X|\mathcal{H}) = (1/2, 1/2, 2) \qquad \mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = (1/2, 5/4, 5/4),$$

Problem 2. If a random variable has characteristic function $\varphi(t) = (1 - |t|)^+$, find its density.

Solution. Use the inverse fourier transform formula. Since φ has bounded support,

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-1}^{1} e^{-itx} \varphi(t) dt \\ &= \frac{1 - \cos(x)}{\pi x^2}. \end{aligned}$$

Problem 3. If X_n converge in distribution to X, and Y_n converge in probability to 0, show that $X_n + Y_n$ converge in distribution to X.

Solution. Suppose F_X is continuous at t. Then

$$\mathbb{P}(X_n + Y_n \le t) \ge \mathbb{P}(X_n \le t - \varepsilon, |Y_n| \le \varepsilon)$$

Since $\mathbb{P}(|Y_n| \le \varepsilon) \to 1$, for any δ eventually, $\mathbb{P}(X_n + Y_n \le t) \ge \mathbb{P}(X_n \le t - \varepsilon) - \delta$. We can take ε so that $\mathbb{P}(X_n \le t - \varepsilon) \ge \mathbb{P}(X \le t) - \delta$, so eventually $\mathbb{P}(X_n + Y_n \le t) \ge F(t) - 2\delta$.

In a similar way, eventually $\mathbb{P}(X_n + Y_n \le t) \le F(t) + 2\delta$.

Problem 4. (a) If $X_n \to X$ and $Y_n \to Y$ in probability show that $X_n + Y_n \to X + Y$ in probability.
(b) Do the same for convergence in L^p for $p \ge 1$.
(c) Show this fails for convergence in L^p for $p < 1$.
(d) Show this fails for convergence in distribution.

Solution. Let $U_n = X_n - X$ and $V_n = Y_n - Y$.
(a) With high probability (i.e. tending to 1) $|U_n| \le \varepsilon$ and similarly for $|V_n|$. Therefore with high probability $|U_n + V_n| \le 2\varepsilon$.
(b) This follows from $\|U_n + V_n\|_p \le \|U_n\|_p + \|V_n\|_p$.
(c) The implication actually holds: For $p < 1$ we have $|U_n + V_n|^p \le |U_n|^p + |V_n|^p$. Take expectations to get the claim.
(d) Let X_n and Y_n all have the same distribution, so they converge in distribution. However, $X_n + Y_n$ might not converge. For example, all are ± 1, and $Y_n = (-1)^n X_n$.

Problem 5. Suppose random variables (X, Y) have joint density $f(x, y)$ on $\mathbb{R}^2$. Let $Z(y) = \int_{\mathbb{R}} x f(x, y) dx / \int_{\mathbb{R}} f(x, y) dx$. Show that $Z = \mathbb{E}(X|Y)$ using the definition.

Problem 6. Construct a martingale X_n so that $X_n \to \infty$ a.s.

Solution. Let the increments be independent with $\mathbb{P}(D_n = 1) = 1 - n^{-2}$ and $\mathbb{P}(D_n = -n^2 + 1) = n^{-2}$. By Borel-Cantelli, eventually $D_n = 1$ so $X_n \to \infty$.

Problem 7. Do not hand this in; a related problem will be in the exam instead. Let X_i be i.i.d. bounded integer random variables, and let $S_n = \sum_{i \le n} X_i$. Suppose $\mathbb{P}(X > 0)$ and $\mathbb{P}(X < 0)$ are both non-zero, and $\mathbb{E}X \neq 0$.

(a) Show that there is some $0 < q \neq 1$ so that q^{S_n} is a martingale.
(b) Find that q when X_n is ± 1 with probabilities $p, 1 - p$.
(c) Use this to find $\mathbb{P}_k(T_n < T_0)$, where $T_a = \inf\{t : X_t = a\}$.

****Problem 8.** If X_n are independent random variables such that $\sum \mathbb{E}X_n$ exists, and that $\sum X_n$ converges a.s., must it be that $\mathbb{E}\sum X_n = \sum \mathbb{E}X_n$?