Exercises

0. The grandfather graph G is as follows. Start with a 3-regular tree $\mathbb{T}_3$ with all edges directed towards some end. Add an edge from each vertex to its grandfather and forget the orientations.

   Prove this is not unimodular: There is an isometry invariant $f(x,y)$ with $\sum_x f(o,x) \neq \sum_x f(x,o)$

1. For iid coins on $\mathbb{Z}$ with $P(\text{Heads}) = \frac{1}{3}$, construct a partition of $\mathbb{Z}$ into triplets of $\{T, T, H\}$, in an invariant way.

   (invariant: If the coins are shifted, so are the groups).

1': Generalize this.

2. For iid coins on $\mathbb{Z}$ with any $p \in (0,1)$ find a fractional matching (mass trasport rule) with $\sum_n f(o,n) = 1$ and $\sum_n f(n,o) = p^{-1} \mathbb{1}_{X_0 = H}$. The rule f should be translation invariant.

3. Do the same on any Cayley graph.

4. Given Poisson processes in $\mathbb{R}^d$ of intensities $\lambda_1 \geq \lambda_2 \geq \lambda_3$ prove that (there exists an invariant matching between the points s.t. every pt is matched to a different colour) iff $\lambda_1 \leq \lambda_2 + \lambda_3$.

5. Construct a 2-col matching on $\mathbb{R}^d$ as follows.
There will be stages $k \in \mathbb{N}$. In stage k split $\mathbb{R}^d$ into boxes of side length 2^k, with random shift (for trans. inv.) points in each box that are not previously matched are paired as much as possible.
Prove that all points are eventually paired (assume $\lambda_1 = \lambda_2$).
Prove that the density of points with $|x - M(x)| > \ell$ is $\le c\ell^{-d/2}$ and deduce $\mathbb{P}(X > \ell) \lesssim \ell^{-d/2}$.

6. Fill in the details in a proof that if a pt. proc. has distinct distances and no ∞ descending chains (a seq. X_n with $|X_n - X_{n+1}| < |X_{n-1} - X_n| \ \forall n$) then there is a unique stable matching.

7. Prove that any sets of n red and n green pts in $\mathbb{R}^2$ with no 4 in a line have a non-crossing matching.