Branching random walk with stretched exponential tails

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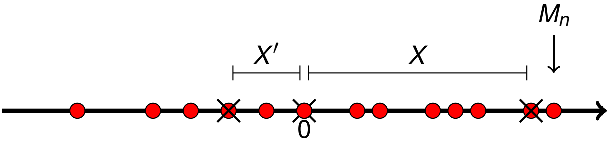
June 5, 2020



- the particles reproduce according to a Galton-Watson process with reproduction mean $m > 1$
- the displacements are iid copies of X such that

$$\mathbb{P}[X > t] \sim e^{-t^r}, \quad r \in (0,1)$$

- $M_n =$ the position of the rightmost particle

Theorem (P.D, N. Gantert, T. Höfelsauer)

Take X such that $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$.
Suppose that (...) and that X has a stretched exponential tail, i.e.

$$\mathbb{P}[X > t] \sim e^{-t^r}, \quad r \in (0,1).$$

Then there are constants γ, σ, α such that: for $r \in \left(0, \frac{2}{3}\right]$,

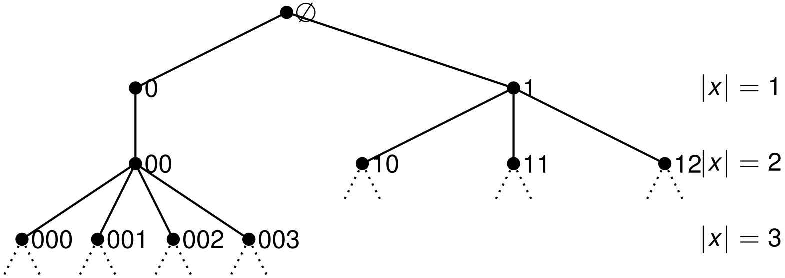
$$\frac{M_n - \alpha n^{\frac{1}{r}}}{\sigma n^{\frac{1}{r}-1}} \to^d F(x) = \mathbb{E}\left[\exp\left\{-\gamma W e^{-x}\right\}\right]$$

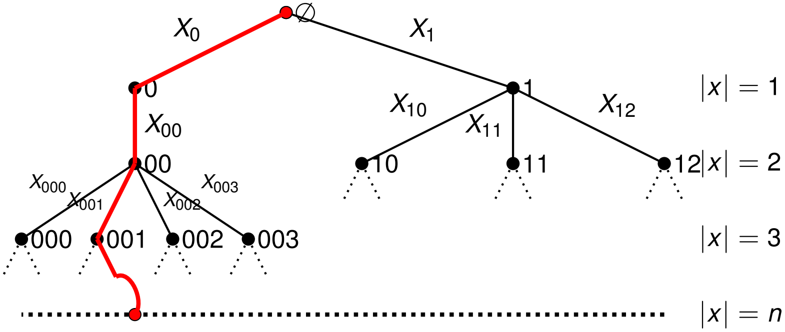
and for $r \in \left(\frac{2}{3}, 1\right)$

$$\frac{M_n - \alpha n^{\frac{1}{r}}}{n^{2-\frac{1}{r}}} \to \frac{1}{2\sigma},$$

where W is a martingale limit associated with the underlying Galton-Watson process.

Let (Z_n) be a Galton-Watson process with reproduction mean $m > 1$.
Assume for simplicity $\mathbb{P}[Z_1 = 0] = 0$.
Let $\mathbb{T} \subseteq \bigcup_{n \geq 0} \mathbb{N}^n$ be the corresponding Galton-Watson tree.





X_y, $y \in \mathbb{T}$ iid

For $x \in \mathbb{T}$,

$$V(x) = \sum_{y \leq x} X_y$$

$$M_n = \max_{|x|=n} V(x)$$

$$\mathbb{P}[X > t] \sim e^{-t^r}, \quad r \in (0,1)$$

$$\mathbb{P}\left[X_1 + X_2 > t\right] \sim \mathbb{P}\left[\max\{X_1, X_2\} > t\right]$$

$S_n = \sum_{1\le k\le n} X_k$, $X_n^* = \max_{1\le k\le n} X_k$

$$\mathbb{P}\left[S_n > t_n\right] \sim ?$$

$$\begin{aligned}\mathbb{P}\left[S_n > cn^{\frac{1}{r}}\right] &\approx \sup_s \mathbb{P}\left[X_n^* > cn^{\frac{1}{r}} - s\right]\mathbb{P}[S_{n-1} > s] \\ &\approx \sup_s n\exp\left\{-\left(cn^{\frac{1}{r}} - s\right)^r - \frac{s^2}{2n}\right\} \\ &\approx \mathbb{P}\left[X_n^* > cn^{\frac{1}{r}} - rc^{r-1}n^{2-\frac{1}{r}}\right]\mathbb{P}\left[S_{n-1} > rc^{r-1}n^{2-\frac{1}{r}}\right]\end{aligned}$$

$$\mathbb{P}\left[S_n > cn^{\frac{1}{r}}\right] \sim n\exp\left\{-c^r n + O\left(n^{3-\frac{2}{r}}\right)\right\}$$

If $|x| = n$, then

$$V(x) \overset{d}{=} S_n = \sum_{k=1}^{n} X_k.$$

$$\begin{aligned}
\mathbb{P}\left[M_n > cn^{\frac{1}{r}}\right] &= \mathbb{P}\left[\exists\, |x| = n,\ V(x) > cn^{\frac{1}{r}}\right] \\
&= \mathbb{P}\left[\sum_{|x|=n} \mathbb{1}_{\left\{V(x) > cn^{\frac{1}{r}}\right\}} \geq 1\right] \\
&\leq \mathbb{E}\left[\sum_{|x|=n} \mathbb{1}_{\left\{V(x) > cn^{\frac{1}{r}}\right\}}\right] = \mathbb{E}[Z_n]\mathbb{P}\left[S_n > cn^{\frac{1}{r}}\right] \\
&= m^n \mathbb{P}\left[S_n > cn^{\frac{1}{r}}\right] \\
&\sim m^n \exp\left\{-c^r n(1 + o(1))\right\}
\end{aligned}$$

$$\frac{M_n}{n^{\frac{1}{r}}} \to \alpha = \log(m)^{\frac{1}{r}}$$

The first term in the asymptotic expansion of M_n is related to the biggest displacement, i.e.

$$N_n = \max_{|y| \leq n} X_y.$$

Proposition

$$\frac{N_n - \alpha n^{\frac{1}{r}}}{\sigma n^{\frac{1}{r}-1}} \to^d H \stackrel{d}{=} \mathbb{E}[\exp\{-\gamma' W e^{-x}\}]$$

Proof.

$x_n \to \infty$

$$\mathbb{P}[N_n \leq x_n] = \mathbb{E}\left[\,\mathbb{P}[N_n \leq x_n \mid (Z_n)\,]\,\right] = \mathbb{E}\left[\mathbb{P}[X \leq x_n]^{Y_n}\right]$$
$$\sim \mathbb{E}\left[\exp\{-Y_n \mathbb{P}[X > x_n]\}\right]$$

$Y_n = \#\{|x| \leq n\} = \sum_{k=1}^{n} Z_k.$

Proof continued .

$m^{-n}Z_n$ is a positive martingale

$$\lim_{n\to\infty} \frac{Z_n}{m^n} = W > 0$$

$$Y_n \sim \frac{m}{m-1}\, W\, m^n$$

$x_n = \alpha n^{\frac{1}{r}} + \sigma n^{\frac{1}{r}-1} x$

$$\begin{aligned} Y_n \mathbb{P}[X > x_n] &\sim \frac{m}{m-1}\, W\, m^n \exp\left\{-\left(\alpha n^{\frac{1}{r}} + \sigma n^{\frac{1}{r}-1} x\right)^r\right\} \\ &\sim \frac{m}{m-1}\, W\, m^n \exp\left\{-\alpha^r n - \alpha^{r-1} r \sigma x\right\} \\ &\sim \gamma'\, W\, \exp\{-x\} \end{aligned}$$

$$\sigma = \frac{\alpha^{1-r}}{r} \quad \gamma' = \frac{m}{m-1}$$

$$\mathbb{P}\left[\frac{N_n - \alpha n^{\frac{1}{r}}}{\sigma n^{\frac{1}{r}-1}} \le x\right] = \mathbb{P}[N_n \le x_n] \to \mathbb{E}[\exp\{-\gamma' W e^{-x}\}]$$

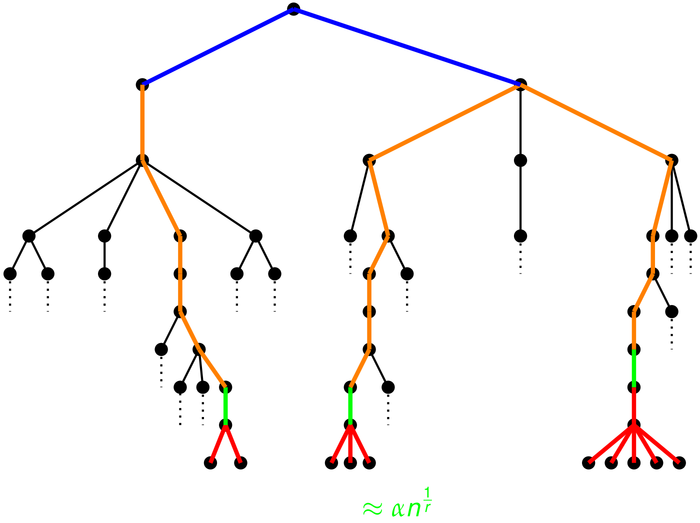
$$\frac{M_n}{n^{\frac{1}{r}}} \to \alpha \qquad \frac{N_n}{n^{\frac{1}{r}}} \to \alpha$$

$$\max_{|x|=n} \left\{ V(x) = \sum_{v \leq x} X_v \; : \; X_v \ll \alpha n^{\frac{1}{r}} \right\} = o(\ldots)$$

$$\#\{|v| \leq n\} = Y_n \approx m^n$$

$$\# \left\{ |v| \leq n \; : \; X_v \approx \alpha n^{\frac{1}{r}} \right\} \approx m^{o(n)}$$

$\left\{ |v| \leq n \; : \; X_v \approx \alpha n^{\frac{1}{r}} \right\}$ is a small subset of $\{|v| \leq n\}$



$$V(x) = R_1(x) + V_0(x) + N(x) + R_2(x)$$
$$= V_0(x) + N(x)$$

$$M_n \approx \max\{V_0(x) + N(x)\}$$

$$N(x) \approx \alpha n^{\frac{1}{r}}$$

$$V_0(x) \stackrel{d}{=} S_{n-o(n)}\Big| X_k \ll \alpha n^{\frac{1}{r}}$$

The contribution of $V_0(x)$ is at most

$$V_0(x) = O\left(n^{2-\frac{1}{r}}\right)$$

on the other hand

$$N_n = \max_{|v| \le n} X_v \stackrel{d}{=} \alpha n^{\frac{1}{r}} + \sigma n^{\frac{1}{r}-1} H(1 + o(1))$$

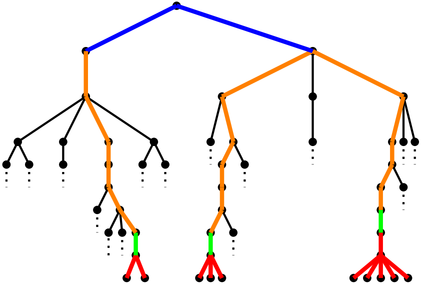
If $r < \frac{2}{3}$, $n^{2-\frac{1}{r}} = o\left(n^{\frac{1}{r}-1}\right)$ and so

$$M_n \approx N_n \stackrel{d}{=} \alpha n^{\frac{1}{r}} + \sigma n^{\frac{1}{r}-1} H(1 + o(1))$$

$$\frac{M_n - \alpha n^{\frac{1}{r}}}{\sigma n^{\frac{1}{r}-1}} \sim \frac{N_n - \alpha n^{\frac{1}{r}}}{\sigma n^{\frac{1}{r}-1}} \to^d H \stackrel{d}{=} \mathbb{E}[\exp\{-\gamma' W e^{-x}\}]$$

$$\frac{M_n - N_n}{n^{\frac{1}{r}-1}} \to 0$$

If $r > \frac{2}{3}$, $n^{\frac{1}{r}-1} = o\left(n^{2-\frac{1}{r}}\right)$



$\approx \alpha n^{\frac{1}{r}}$

$$M_n \approx \max\left\{V_0(x) + N(x)\right\}$$

$$\frac{M_n - \alpha n^{\frac{1}{r}}}{n^{2-\frac{1}{r}}} \to \frac{1}{2\sigma}$$

If $r = \frac{2}{3}$, $n^{2-\frac{1}{r}} = n^{\frac{1}{r}-1} = \sqrt{n}$

$$M_n \approx \max\{N(x) + V_0(x)\}$$

$\mathcal{N}_x \stackrel{d}{=} \Phi(\cdot) \stackrel{d}{=} \mathcal{N}(0,1)$.

$$\frac{M_n - \alpha n^{\frac{1}{r}}}{\sigma\sqrt{n}} \approx \max\left\{\frac{N(x) - \alpha n^{\frac{1}{r}}}{\sigma\sqrt{n}} + \frac{1}{\sigma}\mathcal{N}_x\right\}$$

$$\begin{aligned}
\mathbb{P}\left[\frac{M_n - \alpha n^{\frac{1}{r}}}{\sigma\sqrt{n}} \leq t\right] &\approx \mathbb{E}\left[\mathbb{P}\left[\frac{N(x) - \alpha n^{\frac{1}{r}}}{\sigma\sqrt{n}} + \frac{1}{\sigma}\mathcal{N} > t\right]^{Y_n}\right] \\
&\approx \mathbb{E}\left[\exp\left\{\gamma' W \int e^{-y}(1 - \Phi(\sigma(t-y))dy\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{-\gamma W e^{-t}\right\}\right]
\end{aligned}$$

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and for $r \in \left(\frac{2}{3}, 1\right)$

$$\frac{M_n - \alpha n^{\frac{1}{r}}}{n^{2-\frac{1}{r}}} \to \frac{1}{2\sigma}$$

where W is a martingale limit associated with the underlying Galton-Watson process.

References I

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