Large deviations for random networks & applications

$G \sim G(n,p)$ a random graph on n vertices where every edge occurs ind. wp p.

$H = (V(H), E(H))$

- Take H to be K_3 $\triangle$

X_H - no of copies of H in G.

$E(X_H) \approx n^3 p^3$

$$P(\underbrace{X_H \geq (1+\delta) E(X_H)}_{A}) \qquad \delta > 0.$$

- Infamous upper tail problem. (Janson - Rucinski '02).

- Geometric question

What does the random graph G look like given the event A.

Fact : X_H is a polynomial of independent Bernoulli variables.

Recall some classical concentration & large deviations results for linear functions

Azuma-Hoeffding inequality

$X_1, X_2, \ldots X_n$ are independent mean zero random variables such that $a_i \le X_i \le b_i$ almost surely.

$$S_n = \sum_{i=1}^{n} X_i$$

- $P(|S_n| > t)$

- strategy is to compute exponential moments & then apply Markov.

- $$E(e^{\theta X_i}) \le e^{\theta^2 \underbrace{(b_i - a_i)^2}_{c_i}/8}$$

$$E(e^{\theta S_n}) \le e^{\theta^2 \sum_{i=1}^{n} c_i^2/8}$$

$$P(S_n > t) \le e^{\theta^2 \sum c_i^2/8 - \theta t}. \quad \forall \theta.$$

- optimize over θ.

- X_i are coin tosses.

$X_i \overset{iid}{\sim} Ber(p)$ $\qquad q > p$

$$S_n = np + O(\sqrt{n})$$

$$P(S_n > nq)$$

$$\Lambda(\theta) = \log\left(pe^{\theta} + (1-p)\right)$$

$$\leq e^{(n\Lambda(\theta) - n\theta q)}$$

$$\sup_{\theta} \left(\theta q - \Lambda(\theta)\right)$$

$$= I_p(q) = q \log \frac{q}{p} + (1-q)\log\frac{(1-q)}{1-p}$$

$$P(S_n > nq) \leq e^{-n I_p(q)}$$ (error free bound).

- Lower bound. - (Tilting)
- strategy is to do a change of measure which makes the atypical event typical.

- To get back to the original measure estimate the R-N derivative between the two measures

$$P(A) = \int_A e^{\log \frac{dP}{dQ}} dQ$$

Recall X_H is the number of copies of H in G.

$$t(H,G) = \frac{1}{n^{V(H)}} \sum_{i_1, i_2 \ldots i_k} \prod_{(x,y) \in E(H)} a_{i_x i_y}$$

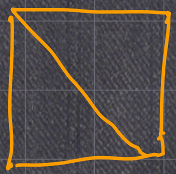
G graph on n vertices with adjacency matrix $A = (a_{ij})$

- What does $G(n,p)$ look like give $t(H,G)$ is large.

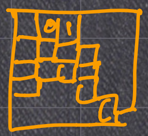
  - $G(n,p)$ continues to look like an Erdos-Renyi type graph but with different densities. (inhomogeneous random graph)

It will be convenient to define a metric on graphs & embed them in the same space.

<u>Defn</u> Let $\mathcal{W}$ be the set of all sym. mble function from $[0,1]^2 \to [0,1]$
— Graphons.



Note that any finite graph naturally embeds in $\mathcal{W}$ as a $\{0,1\}$ valued step function.



For $f, g \in \mathcal{W}$

$$d_{\square}(f,g) = \sup_{S,T \subset [0,1]} \left| \int_{S \times T} (f-g) \right|$$

cut distance.

- Since we don't care about labels of the graphs, we should identify graphs/graphons which are the same upto a relabelling.

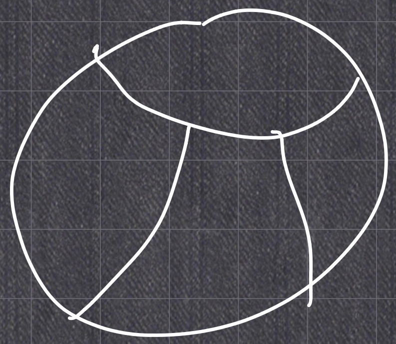
σ - measure pres. bijection on $[0,1]$

$f \sim g$ are equivalent if

$\exists\ \sigma$ st $\quad d_{\square}(f, g \circ \sigma) = 0$

- We would work with the quotient space $\widetilde{\mathcal{W}}$.

$$\begin{array}{c|cccc|}
A_1 & p & p & p & p \\
A_2 & p & p & p & p \\
A_3 & p & p & p & p \\
A_4 & p & p & p & p
\end{array}$$



$G \sim G(n,p)$

$d_{\square}(G, p)$ is typically small.

- A related question is
what is the prob the graph looks like

$$\begin{array}{c|c|c|c|c|}
- A_1 & p_1 & p_2 & p_3 & p_4 \\
A_2 & p_2 & p & p & p \\
A_3 & p_3 & p & p & p \\
A_4 & p_4 & p & p & p
\end{array}$$

- Notice that this is exactly the coin tossing problem.

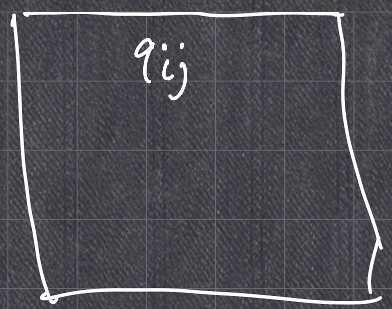
The prob of this, using the same reasoning as the coin tossing is

$$e^{-\left[\binom{A_1}{2} I_p(P_1) + |A_1||A_2| I_p(P_2) \cdots \right]}$$

- We want to find the best possible block graph which makes the atypical event typical.

$$\phi(H, n, p, \delta) = \min\left(\sum_{1 \le i < j \le n} I_p(q_{ij}) : \right.$$

$$t(H, Q) \ge (1+\delta)\, E_p(t(H, G))$$



$Q = (q_{ij})$
is the new
weighted graph.

- $\phi(H, n, p, \delta)$ is the best prob

e

one can get by the strategy of considering inhomogeneous random graphs.

- How to prove that this is indeed the optimal strategy.

- Szemeredi's regularity Lemma

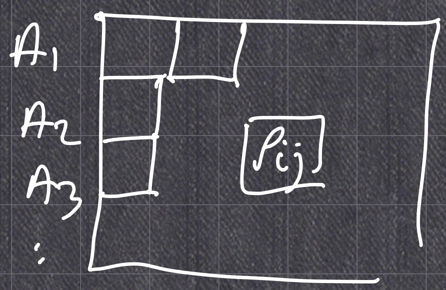
- roughly this says "any" graph can be approximated by a block random graph where the no of blocks is only a function of the error and not the size of the graph.

- (Weak reg. Lemma - Frieze & Kannan)

- Given any graph $G = (V, E)$ there exists a partition of V into

K classes $A_1, A_2, \ldots A_K$ - ρ
such that

. - - .



$$\rho_{ij} = \frac{E_G(A_i, A_j)}{|A_i||A_j|}$$

$$d_\square(G, G_\rho) \leq O\left(\frac{1}{\sqrt{\log K}}\right)$$

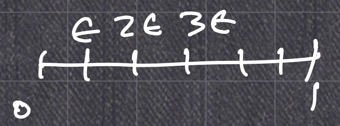
One crucial prop of the cut dist
- (Counting Lemma)

Fix H, and graphons f, g

$$|t(H,f) - t(H,g)| \leq O_H(d_\square(f,g))$$

- Using the above two facts one can compute the probability that G looks like a given block graph and then union bound over all possible choices of block graphs (consider all possible partition of V into K blocks and all possible edge densities

β_{ij} upto an ϵ error)



- If the union bd is over a not too big set, the upper bd one gets is

$$e^{-\underline{\phi(H,n,p,\delta)}+\text{smaller order}}$$

- This fails if p is going to zero with n faster than a poly log.
- Full LDP on graphons for a fixed p was proven by Chatterjee & Varadhan (2011)
- The argument above which is more combinatorial. – Lubetzky-Zhao (2015)).