If H is Δ-regular.

$p \gg \frac{\text{polylog}}{n^{2/\Delta}}$ (Basak–Basu)

following (Harel–Mousset–Samotij) who settled the prob for cliques

beyond this random variables start looking Poisson & that govern LDP.

Lecture 3 07/17

- Talk about how to prove the validity of the variational problem for LDP.
- How to study LDP for spectral statistics.

- Recall.

$\phi(H,n,p,\delta)$ – optimal entropy cost among inhomo. random graphs

$p \gg n^{-1/\Delta}$

$$\lim \frac{\phi(H,n,p,\delta)}{n^2 p^\Delta \log(1/p)} = \begin{cases} \min(\theta, \frac{1}{2}\delta^{2/|V(H)|}) & H \text{ reg} \\ \theta & H \text{ irr} \end{cases}$$

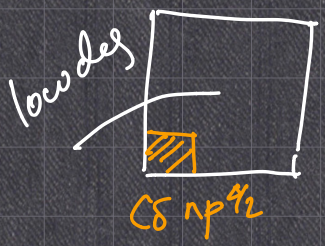
Δ - max degree of H

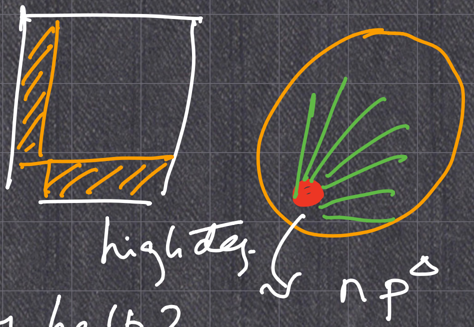
θ - solves

$I_{\ldots}(\theta) = 1+\delta$

$\underbrace{H(\cdot)}$

independence poly of H
res. to max deg vertices





- Why does sparsity help?

For dense graphs we could not say such structural things.

It is an entropy-min problem.

$$\sum_{ij} I_p(q_{ij}) \qquad \otimes \longrightarrow \text{conditions.}$$

when $p \to 0$ $\quad I_p(p+x)$ admits nice poly approx.

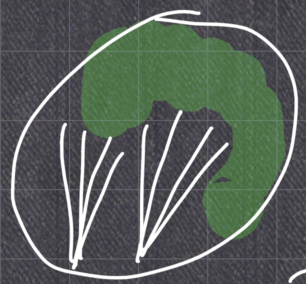
$$I_p(p+x) \gtrsim x^2 \log\left(\frac{1}{p}\right)$$

— To find asym of $\underset{\text{min}}{\underline{\phi(\cdot \cdots)}}$

Upper bd - comes from construction.
clique - anti-clique / hub

- lower bd (challenging).

high level - divide the graph into high degree & low deg vertices



- consider the contribution to the subgraph density from the two parts.

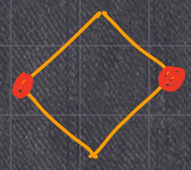
If H is connected. then it's more optimal to just have one part.

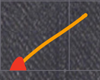
- either the high deg dominates
  — hub
- H is dis conn. then one can have mixtures.
- This indicates the following germ fact is true.

conditioned on high density $t(H, G)$ it is likely that G has a clique or an anti-clique close to the sizes mentioned.

— (Harel, Mousset, Samotij) prove a similar statement, cliques. / Basak, Basu settle the prob for all regular H).

- for sparse p homomorphism & isomorphs – behave differently

 $\approx n^4 p^4$

 $\approx n^3 p^2$

$$n^3 p^2 \gg n^4 p^4$$

$$\frac{1}{n} \gg p^2 \qquad p \ll \frac{1}{\sqrt{n}}$$

- Recall that in the dense case, we approx any graph by a block graph (with no of blocks not growing with n)
  - used coin tossing prob to bound the prob of looking like a given block graph, followed by union bd.

- Strategy: To come up with more efficient covers of the space.
- Possible general approach: Gibbs measure framework / compute exp moments.

Chatt.-Dembo 2014
- Eldan 2018
- Augeri

$$f : \{0,1\}^n \longrightarrow \mathbb{R}$$

$$\pi(x) \propto \frac{e^{f(x)}}{Z} \qquad Z = \frac{1}{2^n} \sum e^{f(x)}$$

Goal: Approx Z. $\qquad P(A) = Z$ if $f = 0$ on A, $= -\infty$ off A.

Gibbs variational principle

$$\log Z = \sup_{\nu} \left(E_{\nu} f - I_{\mu}(\nu) \right) \qquad (\mu = \text{uniform measure})$$

any measure on $\{0,1\}^n$

Exercise : Prove it (Hint: optimal sol. $\nu \propto e^f$.

Mean field approx. Restrict ν to product measures (every co-ordinate is independent)

Exercise - f is linear then the optimal solution is a prod. measure.

$f = \sum a_i x_i$

- Under what conditions is the mean-field strategy reason.

Chatterjee-Dembo

- low gradient complexity - $(\nabla f(x) : x \in \{0,1\}^n)$ has small covering no.

- Second order smooth. $\|f_{ij}\|_\infty$

- Eldan small mean gaussian width

X no second order condition

$$G(K) = E \sup_{x \in K} \langle x . G \rangle$$

G standard gaussian vector.

- Augeri subsequently used convex

analysis to get further improvements.

- Cook-Dembo took a more direct approach to cover graphs by block graphs using spectral properties.

Starting point : P_p is the prod Ber(p) meas.

For any closed convex set $K \in \mathbb{R}^n$

$$P_p(K) \leq e^{-I_p(K)}$$

$$I_p(K) = \inf_{y \in K} I_p(y)$$

- Strategy is to cover the space upto maybe an exceptional set by closed convex sets $\{K_i\}_{i \in I}$ with the add. prop that the r v of interest X does not oss. too much on K_i for any i.

Using spectral prop of adjacency matrices, a cover in the operator norm suffices for K_3.

- obtained essentially by

a spectral proj argument
- For AP of length 3, one can construct covers using connections to Fourier analysis.
- (H, M, S, B-B) cover graphs according to presence of combinatorial str. "seeds" or "cores".

Spectral statistics

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots$ are the eigenvalues of $A(G)$ - adjacency matrix of G.

When p not too small $\lambda_1 \approx np \qquad \lambda_2 \approx \sqrt{np}$

$$P(\lambda_1 > (1+\delta)np) \qquad \lambda_1(A-B)$$

deterministic

- Cook-Dembo proved validity of mean field approach.
- (Bhatt., G.)

$A \quad \lambda_1 > (1+\delta)np$

$$\Rightarrow \quad t(C_s, G) \geq (1+\delta)^s n^s p^s$$

cycle of size s.

Using previous work (BGLZ)

For ...

$$\frac{-\log P(\quad A\quad)}{n^2 p^2 \log(1/p)} = \min\Big(\underbrace{\frac{(1+\delta)^2}{2}}_{\text{clique}},\ \underbrace{\delta(1+\delta)}_{\text{hub}}\Big)$$

$$I_{C_s}(x) = I_{C_{s-1}}(x) + \gamma I_{C_{s-2}}(x)$$

$B = p\,\mathbb{1}\mathbb{1}^T$

one can actually also prove sharp LD tails for λ_2

- (B. G) – hubs don't show up.

- When p is very small.

$$np < \sqrt{\frac{\log n}{\log\log n}} \qquad p \approx \frac{\text{poly}\log}{n}$$

edge of the spectrum $\lambda_1, \lambda_2 \ldots$ are governed by high degree vertices leading to localized eigen vectors.

$\dfrac{\log n}{\log\log n}$

For not too small p,
$\lambda_1 \approx np$
because total no of edges $\frac{n^2}{2}p$

d

$\lambda_1 = \sqrt{d}$

when p is very small
$p = \frac{c}{n}$

- (B. Bhatt., S. Bhatt, G)
  we proved a LDP for the edge of the spectrum (both upper & lower tails)

  by reducing to a LDP for the extremal degrees.