OOPS, 1.6.2020

Branching random walks

BRW

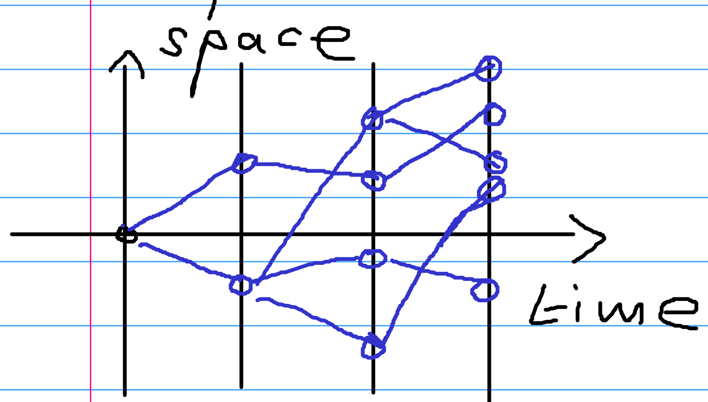
2 ingredients:

- offspring law $p(\cdot)$: $\sum_{k=1}^{\infty} p(k) = 1$, assume $\sum_{k=1}^{\infty} k p(k) > 1$, $\sum_{k=1}^{\infty} k p(k) < \infty$ (often $p(0) = 0$)
- displacement law: RV X, $Var(X) > 0$

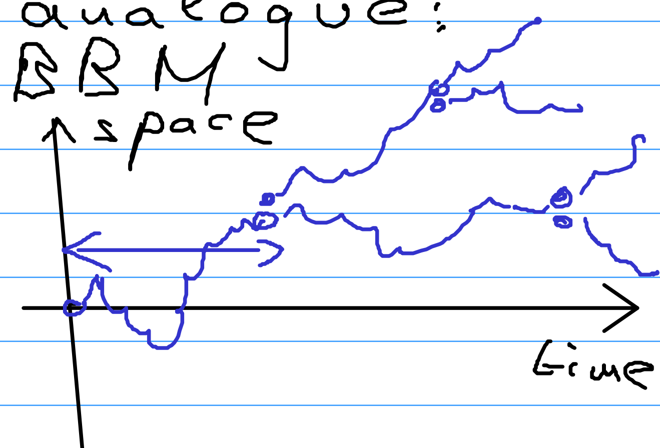
Start with one particle at 0,

- particles produce offspring according to $p(\cdot)$ (and die)
- offspring takes displacements according to X

(all particles behave independently).

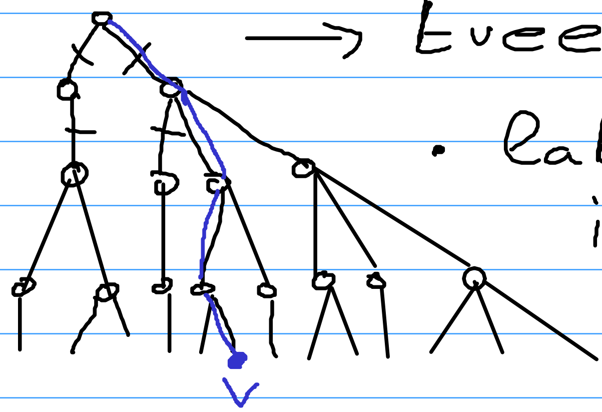


Continuous analogue: BBM



Tree-indexed RW

- GW-process according to $p(\cdot)$:



$\longrightarrow$ tree T

- label edges with iid RV distributed as X

$D_n = \{$vertices in generation $n\}$

$|D_n| =$ Galton-Watson process

$S_v = \sum_{e \in [0,v]} X_e$ position of particle v.

Recall

Thm $m := \sum_{k=0}^{\infty} k p(k) > 1$

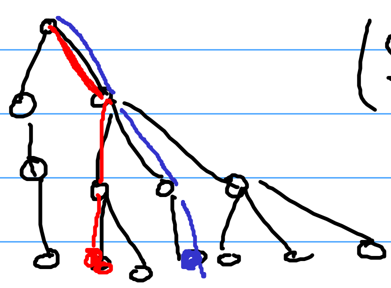
$\Longrightarrow \mathbb{P}[T \text{ infinite}] > 0$

m "reproduction number"

If $p(0) > 0$, can look

$\mathbb{P}^*[\cdot] = \mathbb{P}[\cdot \mid |D_n| > 0, \forall n]$

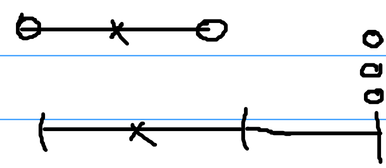
Now:



$(S_v)_{v \in D_n}$ are RV which are not

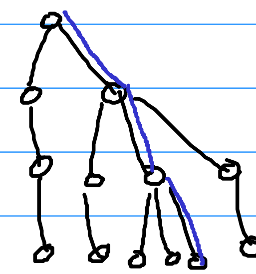
More general model: | independent.

particles produce "offspring + displacements" at once according to some point process.

Example:  each with prob. $\frac{1}{2}$

(dependence between siblings!)

Q $\quad M_n = \sup_{v \in D_n} S_v$



Assume
$$E[e^{\lambda X}] < \infty \text{ for some } \lambda > 0$$

Define $I(y) = \sup_\lambda [\lambda y - \log E[e^{\lambda X}]]$

l.d. rate fct.: for $y > E[X]$,
$$\frac{1}{n} \log P[S_n \geq ny] \longrightarrow -I(y)$$

Indeed,
$$P[S_n \geq ny] \leq E[e^{\lambda S_n}] e^{-\lambda n y}$$
$$\underset{\lambda = \bar{\lambda} \text{ optimal.}}{=} e^{-nI(y)}$$

Define

$$x^* = \sup\{ s \geq \mathbb{E}[X] : I(s) \leq \log m \}$$

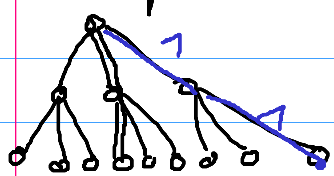
Thm 1 Biggins, Hammersley, Kingman

$$\frac{M_n}{n} \longrightarrow x^* \quad \mathbb{P}^*\text{-a.s.}$$

Exercises:

(i) $X \overset{d}{=} N(0,1)$ Compute x^*

(ii) $p(3) = 1$, $\mathbb{P}[X=0] = \frac{1}{2} = \mathbb{P}[X=1]$



Compute x^*

Do we have $\mathbb{P}[M_n = n, \forall n] > 0$?

(iii) Same as (ii) if $p(2) = 1$.

Intuition:

At time n, have $\approx m^n$ particles

For each $v \in D_n$, $\mathbb{P}[S_v \geq ny] \approx e^{-nI(y)}$

$$e^{-nI(y)} \underbrace{e^{n \log m}}_{m^n} \overset{!}{=} 1 \Rightarrow y = x^*.$$

Proof

(i) First moment method

$$\mathbb{P}[M_n \geq ny] \leq \mathbb{E}\Big[\sum_{v \in D_n} I_{\{S_v \geq ny\}}\Big]$$

$$= \underbrace{\mathbb{E}[|D_n|]}_{m^n} \underbrace{\mathbb{P}[S_n \geq ny]}_{\leq e^{-nI(y)}}$$

Hence if $I(y) > \log m$,

$$\sum_n \mathbb{P}[M_n \geq nv] < \infty$$

$$\Longrightarrow \limsup \frac{M_n}{n} \leq y.$$

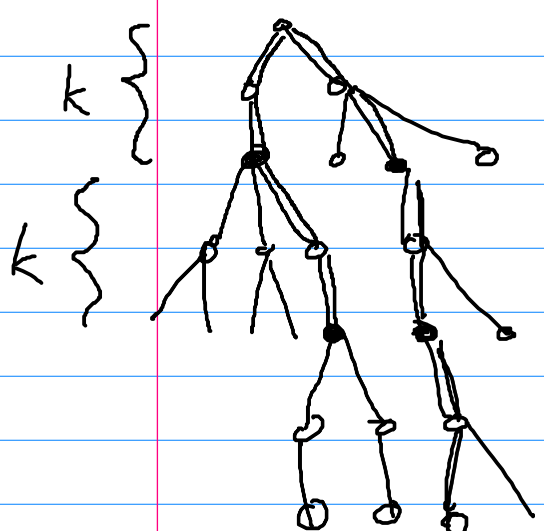
(ii) "Embedded tree"

Assume $y < x^*$

Choose $\varepsilon > 0$ s.t. $I(v) - 2\varepsilon < \log m$

$$\mathbb{P}[S_k \geq ky] \geq e^{-k(I(y)-\varepsilon)}$$

for $k = k_0(\varepsilon)$



"embedded tree":

- keep $v \in D_k$ if $\frac{S_v}{k} \geq y$
- delete v otherwise
- go at level $2k$ etc.

$\longrightarrow$ embedded GW-tree $\tilde{T} \subset T$

What is $\widehat{m}$?

$$\widehat{m} = m^k \mathbb{P}[S_k \geq k\gamma]$$

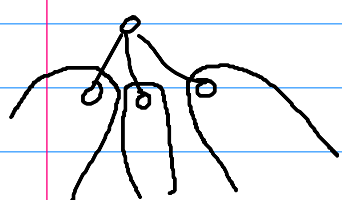
$$\geq e^{k \log m} e^{-k(I(\gamma) - \varepsilon)} \geq e^{\varepsilon k}$$

> 1 for k large enough.

$$\mathbb{P}\left[\liminf_{n \to \infty} \frac{M_n}{n} \geq \gamma\right] > 0.$$

0-1-law for inherited properties

Call a property A of trees inherited if each finite tree has A, and if T has A, then all descendant trees of the children of the root have it.



Then $\mathbb{P}^*[T \text{ has } A] \in \{0,1\}$.

Proof

$$\mathbb{P}[T \text{ has } A] = \mathbb{E}[\mathbb{P}[T \text{ has } A \mid |D_1|]]$$

$$\underset{\text{inherited}}{\leq} \mathbb{E}[\mathbb{P}[T^{(1)} \text{ has } A, \dots, T^{(|D_1|)} \text{ has } A \mid |D_1|]]$$

$$= \mathbb{E}[\mathbb{P}[T \text{ has } A]^{|D_1|}]$$

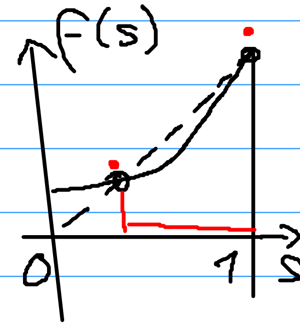
Hence

$$\underbrace{\mathbb{P}[T \text{ has } A]}_{\gamma} \le \underset{q \le \gamma \le f(\gamma)}{f(\mathbb{P}[T \text{ has } A])}$$

where

$$f(s) = \sum_{k=0}^{\infty} s^k p(k) = \mathbb{E}[s^{|D_1|}]$$

On the other hand,



$$\underline{\mathbb{P}[T \text{ has } A] \ge q}$$

$$q = \mathbb{P}[\lim_n |D_n| = 0]$$

$$\Longrightarrow \mathbb{P}[T \text{ has } A] \in \{q, 1\}$$

$$\Longrightarrow \mathbb{P}^*[T \text{ has } A] \in \{0, 1\}.$$

Look at the foll. prop. A:

$$A = \{T \text{ finite}\} \cup \{\liminf_{n \to \infty} \frac{M_n}{n} \le v\}$$

$\blacksquare$